The resistance to a particle of arbitrary shape in translational motion at small Reynolds numbers

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Assuming that the Stokes flow past an arbitrary particle in a uniform stream is known for any three non-coplanar directions of flow, then the force on the body to $O(R)$, for any direction of flow, is given explicitly in terms of these Stokes velocity fields. The Reynolds number (R) based on the maximum particle dimension is assumed small. For bodies with certain types of symmetry it suffices merely to know the Stokes resistance tensor for the body in order to calculate this force. In this case the resulting formula is identical to that of Brenner (1961) and Chester (1962). However, for bodies devoid of such symmetry, their formula is incomplete—there being an additional force at right angles to the uniform stream which remains invariant under a reversal of the flow at infinity. As this additional force is a *lift* force, it follows that the Brenner–Chester formula furnishes the correct *drag* on a body of arbitrary shape; moreover, this drag is always reversed to at least $O(R)$ by a reversal of the uniform flow at infinity.

Exactly analogous formulae are derived using the classical Oseen equations, and it is shown that although this gives both the correct vector force on bodies with the above types of symmetry and the correct *drag* on bodies of arbitrary shape, it gives in general an incorrect *lift* component for completely arbitrary particles.

Finally, the singular perturbation result for the force on an arbitrary body is extended to terms of $O(R^2 \log R)$. This higher-order contribution to the force is given explicitly in terms of the Stokes resistance tensor, and has the property of being reversed by a reversal of the flow at infinity, regardless of the geometry of the body.

These results are collected in the Summary at the end of the paper.

1. Introduction

In a paper by Chester (1962) a formula was given which enables the hydrodynamic force on certain particles immersed in a uniform fluid stream to be determined to terms of $O(R)$, solely from a knowledge of the corresponding Stokes forces. The same formula was originally given in an invariant form by Brenner (1961). However, this formula was proved by Chester only for those bodies whose symmetry is such that a reversal of the uniform flow at infinity reverses the direction of the force on it, without change of magnitude. As pointed out by Chester, the question of whether this formula for the vector force applies to bodies of arbitrary shape has still to be answered.

In the present paper, we shall settle this problem. By using techniques based on singular perturbation methods (Proudman & Pearson 1957) for solving the Navier–Stokes equations at small Reynolds numbers, we shall obtain the force $\mathbf{F}$ on an arbitrary particle as

$$\mathbf{F} = {}^1\mathbf{F} + {}^2\mathbf{F} + \{\mu c U\}\, o(R), \tag{1.1}$$

where ${}^1\mathbf{F}$ is the force given by the formula of Brenner and Chester, and ${}^2\mathbf{F}$ is a purely lift force,‡ at right-angles to the direction of flow at infinity, which remains unaltered by a reversal of flow at infinity.

Whereas ${}^1\mathbf{F}$ can be expressed solely in terms of the Stokes resistance tensor for the body (Brenner 1961, 1963), the computation of ${}^2\mathbf{F}$ requires a detailed knowledge of the complete Stokes velocity field. In particular, the Stokes flow fields must be known for (any) three mutually perpendicular directions of streaming flow past the body. It will be shown that if the body possesses certain symmetry properties, more general than those given by Chester, then ${}^2\mathbf{F}$ will be identically zero. An example will be given of a body for which ${}^2\mathbf{F}$ is non-zero.

In the case of the sphere (Proudman & Pearson 1957) and spheroid (Breach 1961) it has been pointed out that to $O(R)$ the drag determined by the singular perturbation technique is identical to that predicted by the classical Oseen equations, despite the fact that the latter does not furnish the correct asymptotic behaviour of the Navier–Stokes equations to this order in R. In this paper, we shall show that for arbitrary bodies the force to $O(R)$ as determined by the classical Oseen method is

$$\overline{\mathbf{F}} = {}^1\mathbf{F} + {}^2\overline{\mathbf{F}} + \{\mu c U\}\, O(R^2), \tag{1.2}$$

where ${}^1\mathbf{F}$ is the same as that in equation (1.1). ${}^2\overline{\mathbf{F}}$ is a lift force, which in general is non-zero and different from ${}^2\mathbf{F}$. However, for the bodies with the symmetry properties making ${}^2\mathbf{F} = 0$, we have also ${}^2\overline{\mathbf{F}} = 0$, making the correct vector force and the classical Oseen vector force identical in such cases, to $O(R)$.

As ${}^2\mathbf{F}$ and ${}^2\overline{\mathbf{F}}$ are both *lift* forces, it follows that the classical Oseen equations predict the correct *drag* on a body of arbitrary shape, to the first order in R.

We will then obtain the next higher term in the expansion for the force as derived by the singular perturbation technique, i.e. the term of $O(R^2 \log R)$. This extra term will be given in terms of only the Stokes resistance tensor rather than the whole Stokes velocity field. It will be found that this part of the force is reversed by a reversal of flow at infinity, even for an arbitrary body.

2. Fundamental equations

Consider the streaming flow of an incompressible fluid past a stationary solid body of arbitrary shape. Let B' denote the surface of this body. The local fluid motion satisfies the Navier–Stokes and continuity equations

$$\mu \nabla' . \nabla' \mathbf{u}' - \nabla' p' = \rho \mathbf{u}' . \nabla' \mathbf{u}', \quad \nabla' . \mathbf{u}' = 0. \tag{2.1 a, b}$$

‡ ${}^2\mathbf{F}$ does not, however, contain the *entire* lift force on the body; ${}^1\mathbf{F}$ also contains a component at right-angles to the stream velocity vector.

With the exception of constant parameters such as μ and ρ, primed symbols are dimensional and unprimed symbols non-dimensional. For streaming flow with velocity $\mathbf{U}$ (of magnitude U) in the direction $\boldsymbol{\alpha}$ (a unit vector), the boundary conditions are

$$\mathbf{u}' = 0 \quad \text{on} \quad B', \tag{2.2}$$

$$\mathbf{u}' \to U\boldsymbol{\alpha} \quad \text{as} \quad r' \to \infty; \tag{2.3}$$

$\mathbf{r}' = \{r_1', r_2', r_3'\}$ is the position vector of a general point, and $r' = |\mathbf{r}'|$.

Let c be any characteristic body dimension and define the following dimensionless quantities:

$$\{r_1, r_2, r_3\} = c^{-1}\{r_1', r_2', r_3'\}, \tag{2.4}$$

$$p = (c/\mu U)(p' - p_\infty'), \quad \mathbf{u} = \mathbf{u}'/U, \quad R = cU\rho/\mu; \tag{2.5}$$

p_∞' is the constant pressure at infinity. In terms of these quantities the previous equations of motion and boundary conditions become

$$\nabla^2\mathbf{u} - \nabla p = R\mathbf{u}.\nabla\mathbf{u}, \quad \nabla.\mathbf{u} = 0, \tag{2.6a, b}$$

$$\mathbf{u} = 0 \quad \text{on} \quad B, \tag{2.7}$$

$$\mathbf{u} \to \boldsymbol{\alpha} \quad \text{as} \quad r \to \infty. \tag{2.8}$$

3. The inner and outer expansions

The inner expansions are of the form

$$\mathbf{u}(r_1, r_2, r_3) = \mathbf{u}_0(r_1, r_2, r_3) + R\mathbf{u}_1(r_1, r_2, r_3) + o(R), \tag{3.1}$$

$$p(r_1, r_2, r_3) = p_0(r_1, r_2, r_3) + Rp_1(r_1, r_2, r_3) + o(R) \tag{3.2}$$

(see Proudman & Pearson 1957). Upon substituting these into (2.6), (2.7) and equating terms in R^0, one obtains

$$\nabla^2\mathbf{u}_0 - \nabla p_0 = 0, \quad \nabla.\mathbf{u}_0 = 0, \tag{3.3a, b}$$

$$\mathbf{u}_0 = 0 \quad \text{on} \quad B. \tag{3.4}$$

Likewise, equating terms in R^1,

$$\nabla^2\mathbf{u}_1 - \nabla p_1 = \mathbf{u}_0.\nabla\mathbf{u}_0, \quad \nabla.\mathbf{u}_1 = 0, \tag{3.5a, b}$$

$$\mathbf{u}_1 = 0 \quad \text{on} \quad B. \tag{3.6}$$

The conditions imposed on fields $(\mathbf{u}_0, p_0)$, $(\mathbf{u}_1, p_1)$ are insufficient to determine them uniquely. However, additional conditions at $r = \infty$ are furnished by matching the inner and outer expansions.

Dimensionless outer variables $(\tilde{r}_1, \tilde{r}_2, \tilde{r}_3)$ are defined as follows:

$$\{\tilde{r}_1, \tilde{r}_2, \tilde{r}_3\} = R\{r_1, r_2, r_3\}. \tag{3.7}$$

The outer expansions are

$$\mathbf{u}(r_1, r_2, r_3) = \boldsymbol{\alpha} + R\mathbf{U}_1(\tilde{r}_1, \tilde{r}_2, \tilde{r}_3) + o(R), \tag{3.8}$$

$$p(r_1, r_2, r_3) = R^2 P_1(\tilde{r}_1, \tilde{r}_2, \tilde{r}_3) + o(R^2). \tag{3.9}$$

If the ∇ operations in (2.6) are rewritten in terms of outer variables and the outer expansions substituted into the resulting equations, it is found upon equating terms involving like order of R, that $(\mathbf{U}_1, P_1)$ satisfies the equations

$$\tilde{\nabla}^2\mathbf{U}_1 - \tilde{\nabla}P_1 = \boldsymbol{\alpha}.\tilde{\nabla}\mathbf{U}_1, \quad \tilde{\nabla}.\mathbf{U}_1 = 0. \tag{3.10a, b}$$

These are Oseen's equations.

From (2.8) and (3.8) the outer boundary condition satisfied by $\mathbf{U}_1$ is

$$\mathbf{U}_1 \to 0 \quad \text{as} \quad \tilde{r} \to \infty. \tag{3.11}$$

The inner condition imposed on $\mathbf{U}_1$ is dictated by the requirement that the outer and inner expansions be properly matched in their common domain of validity.

4. Zeroth-order inner approximation

By the matching principle, $\mathbf{u}_0$ must agree for large values of r with the leading term of the outer expansion. This requires that

$$\mathbf{u}_0 \to \boldsymbol{\alpha} \quad \text{as} \quad r \to \infty. \tag{4.1}$$

The solution of (3.3), (3.4) and (4.1) is clearly the Stokes solution of the original problem. The force on the body arising from the physical motion $(\mathbf{u}_0', p_0')$ is simply the Stokes force $\mathbf{F}_0$. For convenience we introduce the dimensionless force $\mathbf{f}$ on the body defined by the expression

$$\mathbf{f} = \mathbf{F}/6\pi\mu c U. \tag{4.2}$$

In our analysis we shall require knowledge of the dimensionless Stokes field $(\mathbf{u}_0, p_0)$ at great distances from the body to $O(r^{-2}, r^{-3})$. This may be obtained from Lamb's (1932, pp. 594–7) general solution of the Stokes equations in terms of spherical harmonics as

$$\mathbf{u}_0 = \sum_{n=-\infty}^{+\infty} \left[\frac{1}{2(2n+3)(n+1)}\{(n+3)r^2\nabla p_n - 2np_n\mathbf{r}\} + \mathbf{r}\wedge\nabla\psi_n + \nabla\phi_n\right], \tag{4.3a}$$

$$p_0 = \sum_{n=-\infty}^{\infty} p_n, \tag{4.3b}$$

where p_n, ψ_n and ϕ_n are harmonic functions of degree n in r. Since $\mathbf{u}_0 \to \boldsymbol{\alpha}$, as $r \to \infty$, there can be no term in the expansion (4.3a) of higher degree than r^0, the term of order r^0 being equal to $\boldsymbol{\alpha}$. The first term in the expansion (4.3b) for p_0 must therefore be of $O(r^{-2})$. Now the term of $O(r^{-1})$ in equation (4.3a) for $\mathbf{u}_0$ is $\frac{1}{2}\{r^2\nabla p_{-2} + 4p_{-2}\mathbf{r}\}$. The general form of p_{-2} is $2\boldsymbol{\beta}.\mathbf{r}/r^3$ where $\boldsymbol{\beta}$ is an arbitrary vector. Consider the term of $O(r^{-2})$ in equation (4.3a). This is

$$\tfrac{1}{2}p_{-3}\mathbf{r} + \mathbf{r}\wedge\nabla\psi_{-2} + \nabla\phi_{-1}.$$

Therefore

$$\mathbf{u}_0 = \boldsymbol{\alpha} + \{r^{-1}(\boldsymbol{\beta} + \mathbf{r}.\boldsymbol{\beta}\mathbf{r}/r^2) + \{\tfrac{1}{2}p_{-3}\mathbf{r} + \mathbf{r}\wedge\nabla\psi_{-2} + \nabla\phi_{-1}\} + O(r^{-3}).$$

Since $\mathbf{u}_0 = 0$ on the body and $\nabla.\mathbf{u}_0 = 0$ it follows that $\int_S \mathbf{u}_0.d\mathbf{S} = 0$ for any surface S enclosing the body. Here $d\mathbf{S}' = c^2 d\mathbf{S}$ has the direction of the outer normal to the volume bounded by S. We define ${}_t\mathbf{u}_0$ as the term homogeneous in r^t on the asymptotic expansion of $\mathbf{u}_0$ for large r. Hence

$$\mathbf{u}_0 = {}_0\mathbf{u}_0 + {}_{-1}\mathbf{u}_0 + {}_{-2}\mathbf{u}_0 + O(r^{-3}).$$

Now take S to be a sphere S_L of radius L and centre at the origin. Upon replacing $\mathbf{r}$ by $-\mathbf{r}$, ${}_0\mathbf{u}_0$ and ${}_{-1}\mathbf{u}_0$ remain unaltered, whereas $d\mathbf{S}$ changes sign; hence, it follows that

$$\int_{S_L} ({}_0\mathbf{u}_0).d\mathbf{S} = \int_{S_L} ({}_{-1}\mathbf{u}_0).d\mathbf{S} = 0.$$

Thus letting $L \to \infty$, we find that

$$\lim_{L\to\infty} \int_{S_L} ({}_{-2}\mathbf{u}_0).d\mathbf{S} = 0,\ddagger$$

which requires that

$$\int_{S_L} \{\tfrac{1}{2}p_{-3}\mathbf{r} + \nabla\phi_{-1}\}.d\mathbf{S} = 0.$$

Now, $\phi_{-1} = A/r$, where A is a constant, while $\int_{S_L} \frac{1}{2}p_{-3}\mathbf{r}.d\mathbf{S}$ vanishes since p_{-3} is a spherical harmonic. Hence

$$\int_{S_L} \nabla\phi_{-1}.dS = 0 = -A\int_{S_L} r^2\,dS,$$

which requires that $A = 0$.

The general form of ψ_2 is $\boldsymbol{\gamma}.\mathbf{r}/r^3$ where $\boldsymbol{\gamma}$ is a constant vector. Hence $\mathbf{r}\wedge\nabla\psi_{-2} = \mathbf{r}\wedge\boldsymbol{\gamma}/r^3$. Furthermore, p_{-3} has the general form

$$m_{ij}\frac{\partial^2}{\partial r_i \partial r_j}\left(\frac{1}{r}\right),$$

using the summation convention, where m_{ij} may be taken to be symmetric in i, j and $m_{ii} = 0$. Thus the asymptotic expansion of $\mathbf{u}_0$ is

$$\mathbf{u}_0 = \boldsymbol{\alpha} + \frac{1}{r}\left(\boldsymbol{\beta} + \frac{\mathbf{r}.\boldsymbol{\beta}\mathbf{r}}{r^2}\right) + \left\{\tfrac{1}{2}m_{ij}\frac{\partial^2}{\partial r_i\partial r_j}\left(\frac{1}{r}\right)\mathbf{r} + \frac{\mathbf{r}\wedge\boldsymbol{\gamma}}{r^3}\right\} + O(r^{-3}). \qquad (4.4)$$

We now obtain an alternative form for the $O(r^{-2})$ term in equation (4.4). Taking unit vectors $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ along the axes, define a stokeslet pointing along the r_k-axis at the origin to be that which gives rise to a velocity field of

$$\frac{1}{r}\left(\mathbf{i}_k + \frac{\mathbf{r}.\mathbf{i}_k\mathbf{r}}{r^2}\right) = {}^k\mathbf{u}, \text{ say,}$$

Differentiating this with respect to either r_1, r_2 or r_3, we obtain a term homogeneous in r^{-2} which must satisfy the Stokes equations, and so must be included in the $O(r^{-2})$ term of equation (4.4). Define

$$({}^{kl}\mathbf{u}) = \frac{\partial({}^k\mathbf{u})}{\partial r_l} = \mathbf{i}_l.\nabla({}^k\mathbf{u}).$$

Now $$({}^k\mathbf{u})_j = \frac{1}{r}\left(\delta_{kj} + \frac{r_k r_j}{r^2}\right),$$

therefore $$({}^{kl}\mathbf{u})_j = \frac{\partial}{\partial r_l}\left(\frac{\delta_{kj}}{r} + \frac{r_k r_j}{r^3}\right) = \left\{-\frac{3r_k r_l}{r^5} + \frac{\delta_{kl}}{r^3}\right\}r_j + \frac{\delta_{lj}r_k - \delta_{kj}r_l}{r^3};$$

‡ Actually, any sphere S_L may be used in evaluating this integral since ${}_{-2}\mathbf{u}_0$ is homogeneous in r^{-2} whereas $d\mathbf{S}$ is homogeneous in r^2.

hence $$\tfrac{1}{2}({}^{kl}\mathbf{u}_j + {}^{lk}\mathbf{u}_j) = \left\{-\frac{3r_k r_l}{r^5} + \frac{\delta_{kl}}{r^3}\right\} r_j$$

and $$\tfrac{1}{2}({}^{kl}\mathbf{u}_j - {}^{lk}\mathbf{u}_j) = \frac{\delta_{lj} r_k - \delta_{kj} r_l}{r^3} = \frac{\epsilon_{jpn} r_n \epsilon_{klp}}{r^3};$$

thus $$\tfrac{1}{2}\epsilon_{klm}({}^{kl}\mathbf{u}_j - {}^{kl}\mathbf{u}_j) = \frac{\epsilon_{jpn} r_n \epsilon_{klp}\epsilon_{klm}}{r^3} = \frac{2\epsilon_{jmn} r_n}{r^3}.$$

Now from equation (4.4) we have

$$\begin{aligned}({}_{-2}\mathbf{u}_0)_j &= \tfrac{1}{2}m_{kl}\left(\frac{3r_k r_l}{r^5} - \frac{\delta_{kl}}{r^3}\right) r_j - \frac{\epsilon_{jmn}\gamma_m r_n}{r^3} \\ &= -\tfrac{1}{4}({}^{kl}\mathbf{u}_j + {}^{lk}\mathbf{u}_j)\, m_{kl} - \tfrac{1}{4}\gamma_m \epsilon_{klm}({}^{kl}\mathbf{u}_j - {}^{lk}\mathbf{u}_j) \\ &= {}^{kl}\mathbf{u}_j\{-\tfrac{1}{2}m_{kl} - \tfrac{1}{2}\epsilon_{klm}\gamma_m\} \\ &= \lambda_{pq}({}^{qp}\mathbf{u}_j),\end{aligned}$$

where λ_{pq} is a tensor, *not* in general symmetric.

Write $[\mathbf{s}(\boldsymbol{\alpha})]$ for the velocity field due to a stokeslet of value $\boldsymbol{\alpha}$ at the origin, i.e.

$$[\mathbf{s}(\boldsymbol{\alpha})] = \frac{1}{r}\left(\boldsymbol{\alpha} + \frac{\mathbf{r}\,.\,\boldsymbol{\alpha}\mathbf{r}}{r^2}\right). \tag{4.5}$$

Also let $[t(\boldsymbol{\alpha})]$ be the corresponding pressure field, i.e.

$$[t(\boldsymbol{\alpha})] = \frac{2}{r^3}\boldsymbol{\alpha}\,.\,\mathbf{r}. \tag{4.6}$$

Therefore $$({}^{qp}\mathbf{u}) = \mathbf{i}_p\,.\,\nabla[\mathbf{s}(\mathbf{i}_q)].$$

The asymptotic expansion (4.4) for $\mathbf{u}_0$ now becomes

$$\mathbf{u}_0 = \boldsymbol{\alpha} + [\mathbf{s}(\boldsymbol{\beta})] + \lambda_{pq}\mathbf{i}_p\,.\,\nabla[\mathbf{s}(\mathbf{i}_q)] + O(r^{-3}). \tag{4.7a}$$

The corresponding pressure must therefore be

$$p_0 = [t(\boldsymbol{\beta})] + \lambda_{pq}\mathbf{i}_p\,.\,\nabla[t(\mathbf{i}_q)] + O(r^{-4}). \tag{4.7b}$$

The dimensionless force $\mathbf{f}$ on the body, defined by equation (4.2), can be expressed in the form

$$\mathbf{f} = \mathbf{f}_0 + R\mathbf{f}_1 + o(R), \tag{4.8}$$

where $\mathbf{f}_0$ is the Stokes force on the body due to $(\mathbf{u}_0, p_0)$, and $\mathbf{f}_1$ the force due to $(\mathbf{u}_1, p_1)$.

We will now find the Stokes force $\mathbf{f}_0$ in terms of the coefficients involved in the asymptotic expansions (4.7). If $(p_0)_{ij}$ is the dimensionless stress tensor

$$[(p_0)'_{ij} = (U\mu/c)(p_0)_{ij}],$$

then $(p_0)_{ij,j} = 0$. Integrate this over the volume V_L bounded by the surface B of the body, and the large sphere S_L. Then

$$\int_{S_L}(p_0)_{ij}dS_j + \int_B (p_0)_{ij}dS_j = 0.$$

$\mathbf{f}_0$ is the dimensionless Stokes force on the body and so

$$6\pi(f_0)_i = -\int_B (p_0)_{ij}\,dS_j = +\int_{S_L} (p_0)_{ij}\,dS_j$$
$$= -\int_{S_L} (p_0)\,dS_i + 2\int_{S_L} (e_0)_{ij}\,dS_j, \tag{4.9}$$

where $(e_0)_{ij} = \frac{1}{2}\{(u_0)_{i,j} + (u_0)_{j,i}\}$ is the dimensionless rate-of-strain tensor $[(e_0)'_{ij} = (U/c)\,(e_0)_{ij}]$. As $L \to \infty$, the contributions to the integrals from the terms of order (r^{-2}, r^{-3}) in $(\mathbf{u}_0, p_0)$ tend to zero. Thus the contribution to $\mathbf{f}_0$ comes entirely from the stokeslet term $[\mathbf{s}(\boldsymbol{\beta})]$. On calculating (p_0) and $(e_0)_{ij}$ in spherical polar co-ordinates with axis in the direction $\boldsymbol{\beta}$ and substituting in equation (4.9) we find that

$$\mathbf{f}_0 = -\tfrac{4}{3}\boldsymbol{\beta}. \tag{4.10}$$

Expressing the term $O(r^{-1}, r^{-2})$ in $(\mathbf{u}_0, p_0)$ in terms of $\mathbf{f}_0$ we obtain

$$\left.\begin{aligned} \mathbf{u}_0 &= \boldsymbol{\alpha} - \tfrac{3}{4}[\mathbf{s}(\mathbf{f}_0)] + \lambda_{pq}\mathbf{i}_p\,.\,\nabla[\mathbf{s}(\mathbf{i}_q)] + O(r^{-3}),\\ p_0 &= -\tfrac{3}{4}[t(\mathbf{f}_0)] + \lambda_{pq}\mathbf{i}_p\,.\,\nabla[t(\mathbf{i}_q)] + O(r^{-4}). \end{aligned}\right\} \tag{4.11}$$

In deriving equation (4.11), we have not chosen a particular origin. A change of origin will in general alter the terms of order (r^{-2}, r^{-3}) in $(\mathbf{u}_0, p_0)$, while the terms of order (r^0) (r^{-1}) in $\mathbf{u}_0$ and order (r^{-2}) in p_0 are invariant.

5. First-order outer approximation

Expressing the Stokes solution $(\mathbf{u}_0, p_0)$ in terms of the outer variables we obtain

$$\left.\begin{aligned} \mathbf{u}_0 &= \boldsymbol{\alpha} - \tfrac{3}{4}R[\tilde{\mathbf{s}}(\mathbf{f}_0)] + O(R^2),\\ p_0 &= -\tfrac{3}{4}R^2[\tilde{t}(\mathbf{f}_0)] + O(R^3), \end{aligned}\right\} \tag{5.1}$$

where $[\tilde{\mathbf{s}}(\mathbf{f}_0)]$ and $[\tilde{t}(\mathbf{f}_0)]$ are the values of $[\mathbf{s}(\mathbf{f}_0)]$ and $[t(\mathbf{f}_0)]$ respectively in which $\mathbf{r}$ is replaced by $\tilde{\mathbf{r}}$. Hence for $(\mathbf{U}_1, P_1)$ to be properly matched with the inner expansion, we require

$$\left.\begin{aligned} \mathbf{U}_1 &\simeq -\frac{3}{4\tilde{r}}\left\{\mathbf{f}_0 + \frac{\tilde{\mathbf{r}}\,.\,\mathbf{f}_0\tilde{\mathbf{r}}}{\tilde{r}^2}\right\} \quad \text{as} \quad \tilde{r} \to 0,\\ P_1 &\simeq -\frac{3}{2\tilde{r}^3}\mathbf{f}_0\,.\,\tilde{\mathbf{r}} \quad \text{as} \quad \tilde{r} \to 0. \end{aligned}\right\} \tag{5.2}$$

$(\mathbf{U}_1, P_1)$ satisfy equations (3.10) with boundary conditions (3.11) and (5.2), and it may be shown that these have the solution‡

$$\mathbf{U}_1 = -\frac{3}{4\tilde{r}}\left\{\mathbf{f}_0 + \frac{1}{\tilde{r}^2}(\tilde{\mathbf{r}}\,.\,\mathbf{f}_0\,\tilde{\mathbf{r}})\right\}\exp\left[-\tfrac{1}{2}(\tilde{r} - \tilde{\mathbf{r}}\,.\,\boldsymbol{\alpha})\right]$$
$$+\tfrac{3}{2}\{1 - [1 + \tfrac{1}{2}(\tilde{r} - \tilde{\mathbf{r}}\,.\,\boldsymbol{\alpha})]\exp\left[-\tfrac{1}{2}(\tilde{r} - \tilde{\mathbf{r}}\,.\,\boldsymbol{\alpha})\right]\}\,(\mathbf{f}_0\,.\,\tilde{\nabla})\,\tilde{\nabla}\log(\tilde{r} - \tilde{\mathbf{r}}\,.\,\boldsymbol{\alpha}), \tag{5.3a}$$

$$P_1 = -\frac{3}{2\tilde{r}^3}\mathbf{f}_0\,.\,\tilde{\mathbf{r}}. \tag{5.3b}$$

‡ Throughout the analysis, 'log' refers to the natural logarithm.

The expansions of $\mathbf{U}_1$ and P_1 for small $\tilde{r}$ are thus

$$\left.\begin{aligned} \mathbf{U}_1 &= -\frac{3}{4\tilde{r}}\left\{\mathbf{f}_0+\left(\frac{\tilde{\mathbf{r}}\,.\,\mathbf{f}_0\tilde{\mathbf{r}}}{\tilde{r}^2}\right)\right\}+\tfrac{3}{16}(3\mathbf{f}_0-\boldsymbol{\alpha}\,.\,\mathbf{f}_0\boldsymbol{\alpha}) \\ &\quad+\frac{3}{16\tilde{r}}\left(\boldsymbol{\alpha}\,.\,\mathbf{f}_0\tilde{\mathbf{r}}+\tilde{\mathbf{r}}\,.\,\mathbf{f}_0\boldsymbol{\alpha}-3\boldsymbol{\alpha}\,.\,\tilde{\mathbf{r}}\mathbf{f}_0-\frac{1}{\tilde{r}^2}\boldsymbol{\alpha}\,.\,\tilde{\mathbf{r}}\mathbf{f}_0\,.\,\tilde{\mathbf{r}}\mathbf{r}\right) \\ &\quad+O(\tilde{r}), \\ P_1 &= -\frac{3}{2\tilde{r}^3}\mathbf{f}_0\,.\,\tilde{\mathbf{r}}. \end{aligned}\right\} \tag{5.4}$$

Rearranging terms, and changing to the inner variables we obtain

$$R\mathbf{U}_1 = -\tfrac{3}{4}[\mathbf{s}(\mathbf{f}_0)]+\tfrac{3}{16}R\{\boldsymbol{\alpha}\,.\,\nabla[r(-3\mathbf{f}_0+\mathbf{r}\,.\,\mathbf{f}_0\mathbf{r}/r^2)]+(3\mathbf{f}_0-\boldsymbol{\alpha}\,.\,\mathbf{f}_0\boldsymbol{\alpha})\}+O(R^2). \tag{5.5}$$

The term $-\frac{3}{4}[\mathbf{s}(\mathbf{f}_0)]$ has already been matched on to the zeroth-order inner solution. Also

$$R^2P_1 = -(3/2r^3)\,\mathbf{f}_0\,.\,\mathbf{r}, \tag{5.6}$$

which has similarly been matched already.

6. First-order inner approximation

$(\mathbf{u}_1, p_1)$ are to satisfy the equations (3.5) with the boundary conditions

$$\left.\begin{aligned} &\mathbf{u}_1 = 0 \quad \text{on} \quad B, \\ &\mathbf{u}_1 \sim \tfrac{3}{16}\{\boldsymbol{\alpha}\,.\,\nabla[r(-3\mathbf{f}_0+\mathbf{r}\,.\,\mathbf{f}_0\mathbf{r}/r^2)]+(3\mathbf{f}_0-\boldsymbol{\alpha}\,.\,\mathbf{f}_0\boldsymbol{\alpha})\} \quad \text{as} \quad r\to\infty, \end{aligned}\right\} \tag{6.1}$$

where the second condition is the matching condition derived from equation (5.5). Furthermore, from equation (5.6) we also require that

$$p_1 = o(r^{-1}) \quad \text{as} \quad r\to\infty. \tag{6.2}$$

From the expression for $\mathbf{u}_0$, given in (4.11) we obtain

$$\begin{aligned} \mathbf{u}_0\,.\,\nabla\mathbf{u}_0 = -\tfrac{3}{4}\boldsymbol{\alpha}\,.\,\nabla[\mathbf{s}(\mathbf{f}_0)]+\lambda_{pq}(\boldsymbol{\alpha}\,.\,\nabla)\,(\mathbf{i}_p\,.\,\nabla)\,[\mathbf{s}(\mathbf{i}_q)] \\ +\tfrac{9}{16}[\mathbf{s}(\mathbf{f}_0)]\,.\,\nabla[\mathbf{s}(\mathbf{f}_0)]+O(r^{-4}). \end{aligned} \tag{6.3}$$

Consider the equations

$$\nabla^2\mathbf{u}''-\nabla p'' = [\mathbf{s}(\mathbf{f}_0)], \quad \nabla\,.\,\mathbf{u}'' = 0. \tag{6.4}$$

Since $[\mathbf{s}(\mathbf{f}_0)]$ is axially symmetric about the direction $(\mathbf{f}_0)$, we see that there is a particular integral for $\mathbf{u}''$ also axially symmetric. Thus, by taking spherical polar axes with axis in the direction of $\mathbf{f}_0$, equation (6.4) may be solved in terms of a stream function to give as the particular integral

$$\mathbf{u}'' = -\tfrac{1}{4}r\{-3\mathbf{f}_0+\mathbf{r}\,.\,\mathbf{f}_0\mathbf{r}/r^2\}, \quad p''=0. \tag{6.5}$$

In a similar manner a particular integral of the equations

$$\nabla^2\mathbf{u}'''-\nabla p''' = [\mathbf{s}(\mathbf{f}_0)]\,.\,\nabla[\mathbf{s}(\mathbf{f}_0)], \quad \nabla\,.\,\mathbf{u}''' = 0, \tag{6.6}$$

is

$$\mathbf{u}''' = \tfrac{1}{2}r^{-2}\{\mathbf{f}_0\,.\,\mathbf{r}\mathbf{f}_0-f_0^2\mathbf{r}+2(\mathbf{f}_0\,.\,\mathbf{r})^2\mathbf{r}/r^2\}, \tag{6.7a}$$

$$p''' = r^{-2}\{-f_0^2+2(\mathbf{f}_0\,.\,\mathbf{r})^2/r^2\}. \tag{6.7b}$$

Consider now a general equation of the form

$$\nabla^2\mathbf{u}-\nabla p = \mathbf{g}(\mathbf{r}), \quad \nabla.\mathbf{u} = 0, \tag{6.8}$$

where $\mathbf{g}(\mathbf{r})$ is a given vector field. Then, if $\mathbf{d}$ is a constant vector,

$$\nabla^2(\mathbf{d}.\nabla\mathbf{u})-\nabla(\mathbf{d}.\nabla p) = \mathbf{d}.\nabla[\mathbf{g}(\mathbf{r})], \quad \nabla.(\mathbf{d}.\nabla\mathbf{u}) = 0,$$

and hence $\mathbf{u}^* = \mathbf{d}.\nabla\mathbf{u}$, $p^* = \mathbf{d}.\nabla p$, is a solution of the equations

$$\nabla^2\mathbf{u}^*-\nabla p^* = \mathbf{d}.\nabla[\mathbf{g}(\mathbf{r})], \quad \nabla.\mathbf{u}^* = 0. \tag{6.9}$$

Using this result, together with the solutions (6.5) and (6.7), we may immediately write down a particular integral of the equations (3.5) for $(\mathbf{u}_1, p_1)$ as

$$\begin{aligned}\mathbf{u}_1 = &+\tfrac{3}{16}\boldsymbol{\alpha}.\nabla[r(-3\mathbf{f}_0+\mathbf{f}_0.\mathbf{rr}/r^2)]\\ &-\tfrac{1}{4}\lambda_{pq}(\boldsymbol{\alpha}.\nabla)(\mathbf{i}_p.\nabla)[r(-3\mathbf{i}_q+\mathbf{r}.\mathbf{i}_q\mathbf{r}/r^2)]\\ &+\tfrac{9}{32}r^{-2}\{\mathbf{f}_0.\mathbf{r}\mathbf{f}_0-f_0^2\mathbf{r}+2(\mathbf{f}_0\mathbf{r})^2\mathbf{r}/r^2\}+O(r^{-2}),\end{aligned} \tag{6.10a}$$

$$p_1 = +\tfrac{9}{16}r^{-2}\{-f_0^2+2(\mathbf{f}_0.\mathbf{r})^2/r^2\}+O(r^{-3}). \tag{6.10b}$$

To this must be added a complementary function which to the required order is of the form

$$\mathbf{u}_1 = \boldsymbol{\gamma}+[\mathbf{s}(\boldsymbol{\delta})]+O(r^{-2}), \quad p_1 = [t(\boldsymbol{\delta})]+O(r^{-3}),$$

where $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$ are constant vectors. From the boundary conditions at infinity, we see that

$$\boldsymbol{\gamma} = \tfrac{3}{16}(3\mathbf{f}_0-\boldsymbol{\alpha}.\mathbf{f}_0\boldsymbol{\alpha}). \tag{6.11}$$

Hence the complete solution for $(\mathbf{u}_1, p_1)$ is

$$\begin{aligned}\mathbf{u}_1 = &\tfrac{3}{16}(\boldsymbol{\alpha}.\nabla)[r(-3\mathbf{f}_0+\mathbf{r}.\mathbf{f}_0\mathbf{r}/r^2)]+\tfrac{3}{16}(3\mathbf{f}_0-\boldsymbol{\alpha}.\mathbf{f}_0\boldsymbol{\alpha})\\ &-\tfrac{1}{4}\lambda_{pq}(\boldsymbol{\alpha}.\nabla)(\mathbf{i}_p.\nabla)[r(-3\mathbf{i}_q+\mathbf{r}.\mathbf{i}_q\mathbf{r}/r^2)]\\ &+\tfrac{9}{32}r^{-2}\{\mathbf{f}_0\mathbf{r}\mathbf{f}_0-f_0^2\mathbf{r}+2(\mathbf{f}_0.\mathbf{r})^2\mathbf{r}/r^2\}+[\mathbf{s}(\boldsymbol{\delta})]+O(r^{-2}),\end{aligned} \tag{6.12a}$$

$$p_1 = \tfrac{9}{16}r^{-2}\{-f_0^2+2(\mathbf{f}_0.\mathbf{r})^2/r^2\}+[t(\boldsymbol{\delta})]+O(r^{-3}). \tag{6.12b}$$

Since $p_1 = O(r^{-2})$ as $r\to\infty$, we see that the boundary condition (6.2) is also satisfied. The constant vector $\boldsymbol{\delta}$ is determined by the boundary condition that $\mathbf{u}_1 = 0$ on the surface B of the body.

If $(p_1)_{ij}$ is the dimensionless stress tensor due to $(\mathbf{u}_1, p_1)$ then the equations (3.5) are equivalent to

$$(p_1)_{ij,j} = (u_0)_j(u_0)_{i,j}, \quad (u_1)_{j,j} = 0. \tag{6.13}$$

Since $\nabla.\mathbf{u}_0 = (u_0)_{j,j} = 0$, we may write the latter in the form

$$[(p_1)_{ij}-(u_0)_i(u_0)_j]_{,j} = 0.$$

Integrating this over the volume V_L between B and S_L we obtain

$$\int_{B+S_L}\{(p_1)_{ij}-(u_0)_i(u_0)_j\}\,dS_j = 0.$$

But $\mathbf{u}_0 = 0$ on B and

$$(\mathbf{f}_1)_i = -\frac{1}{6\pi}\int_B(p_1)_{ij}dS_j,$$

hence

$$(f_1)_i = \frac{1}{6\pi}\int_{S_L}(p_1)_{ij}dS_j-\frac{1}{6\pi}\int_{S_L}(u_0)_i(u_0)_jdS_j. \tag{6.14}$$

Thus, in order to calculate $(\mathbf{f}_1)$, $(p_1)_{ij}$ must be known to $O(r^{-2})$ for large values of r. Therefore we need to know $(\mathbf{u}_1, p_1)$ to $O(r^{-1}, r^{-2})$ and in particular the value of $\boldsymbol{\delta}$. However, it will be found that if the Stokes velocity field $\mathbf{u}_0$ is known for any three non-coplanar directions of flow at infinity then, by the use of certain integral relations, the force $\mathbf{f}_1$ due to $(\mathbf{u}_1, p_1)$ (or equivalently the value of $\boldsymbol{\delta}$), can be expressed solely in terms of $\mathbf{u}_0$.

7. First-order force

We now obtain the integral relations mentioned in the preceding paragraph and proceed subsequently to calculate the force $\mathbf{f}_1$.

Let $(\mathbf{u}^*, p^*)$ be the (dimensionless) Stokes velocity field due to a uniform stream of (dimensionless) velocity $\mathbf{V}$ at infinity, $\mathbf{V}$ being arbitrary. Then

$$\left.\begin{array}{c} \nabla^2\mathbf{u}^* - \nabla p^* = 0 \quad \nabla . \mathbf{u}^* = 0; \\ \mathbf{u}^* = 0 \quad \text{on} \quad B, \quad \mathbf{u}^* \to \mathbf{V} \quad \text{as} \quad r \to \infty. \end{array}\right\} \tag{7.1}$$

The equation of motion may be put in the form

$$p^*_{ij,j} = 0, \tag{7.2}$$

where p^*_{ij} is the stress tensor due to $(\mathbf{u}^*, p^*)$. From equation (6.13),

$$(p_1)_{ij,j} = [(u_0)_i (u_0)_j]_{,j}. \tag{7.3}$$

Take the scalar product of (7.3) with u^*_i, and (7.2) with $(u_1)_i$ and subtract to obtain

$$(u^*)_i (p_1)_{ij,j} - (u_1)_i (p^*)_{ij,j} = [(u_0)_i (u_0)_j]_{,j} (u^*)_i. \tag{7.4}$$

Therefore

$$[(u^*)_i (p_1)_{ij} - (u_1)_i (p^*)_{ij}]_{,j} - (u^*)_{i,j} (p_1)_{ij} + (u_1)_{i,j} (p^*)_{ij} = [(u_0)_i (u_0)_j]_{,j} (u^*)_i. \tag{7.5}$$

Since $\nabla . \mathbf{u}_1 = \nabla . \mathbf{u}^* = 0$, it follows (Lamb 1932, p. 617) that

$$(u_1)_{i,j} (p^*)_{ij} = (u^*)_{i,j} (p_1)_{ij}. \tag{7.6}$$

From equations (7.5) and (7.6)

$$[(u^*)_i (p_1)_{ij} - (u_1)_i (p^*)_{ij}]_{,j} = [(u_0)_i (u_0)_j (u^*)_i]_{,j} - (u_0)_i (u_0)_j (u^*)_{i,j}. \tag{7.7}$$

But $$(u_0)_i (u_0)_j (u^*)_{i,j} = \tfrac{1}{2}(u_0)_i (u_0)_j (u^*_{i,j} + u^*_{j,i}) = (u_0)_i (u_0)_j (e^*)_{ij},$$

where e^*_{ij} is the rate of strain tensor for the flow $\mathbf{u}^*$. Therefore

$$[(u^*)_i (p_1)_{ij} - (u_1)_i (p^*)_{ij} - (u_0)_i (u_0)_j (u^*)_i]_{,j} + (u_0)_i (u_0)_j (e^*)_{ij} = 0. \tag{7.8}$$

Now integrate this over the volume V_L bounded externally by the sphere S_L, of radius L, and internally by the surface B of the body, giving

$$\int_{B+S_L} \{(u^*)_i (p_1)_{ij} - (u_1)_i (p^*)_{ij} - (u_0)_i (u_0)_j (u^*)_i\} dS_j + \int_{V_L} (u_0)_i (u_0)_j (e^*)_{ij} dV = 0. \tag{7.9}$$

Since $\mathbf{u}^* = \mathbf{u}_1 = \mathbf{u}_0 = 0$ on B, the surface integrals over B vanish. Thus, if we let $L \to \infty$, equation (7.9) may be written

$$\{(I_1) + (I_4)\} - (I_2) - (I_3) - (I_4) + \int_{\Gamma} (u_0)_i (u_0)_j (e^*)_{ij} dV = 0, \tag{7.10}$$

where Γ is the whole of space external to the surface B, and

$$(I_1) = \lim_{L\to\infty} \int_{S_L} (u^*)_i (p_1)_{ij} dS_j, \tag{7.11}$$

$$(I_2) = \lim_{L\to\infty} \int_{S_L} (u_1)_i (p^*)_{ij} dS_j, \tag{7.12}$$

$$(I_3) = \lim_{L\to\infty} \int_{S_L} (u_0)_i (u_0)_j (u^*)_i dS_j, \tag{7.13}$$

$$(I_4) = \lim_{L\to\infty} -V_i \int_{S_L} (u_0)_i (u_0)_j dS_j. \tag{7.14}$$

Our motivation for introducing the extraneous integral (I_4) will shortly become clear. We now evaluate each of these surface integrals in turn.

Consider (I_1). The terms in the integrand only contribute to (I_1) if they do not tend to zero faster than r^{-2} as $r \to \infty$. Hence since $\mathbf{u}^* = O(1)$ and $\mathbf{u}_1 = O(1)$ as $r \to \infty$, it follows that contributions to (I_1) come only from the terms ${}_0\mathbf{u}^*$ and ${}_{-1}\mathbf{u}^*$, i.e. the terms homogeneous in r^0 and r^{-1} in the asymptotic expansion of $\mathbf{u}^*$ for large r. Writing $({}_{-n}p_1)_{ij}$ as the stress tensor corresponding to the velocity homogeneous in r^{-n} in the asymptotic expansion of $\mathbf{u}_1$, we obtain

$$\begin{aligned}(I_1) &= \lim_{L\to\infty} \int_{S_L} [({}_0u^*)_i + ({}_{-1}u^*)_i] ({}_0p_1)_{ij} dS_j + \lim_{L\to\infty} \int_{S_L} ({}_0u^*)_i ({}_{-1}p_1)_{ij} dS_j \\ &= \lim_{L\to\infty} 2\int_{S_L} \{V_i + [\mathbf{s}(\boldsymbol{\epsilon})]_i\} ({}_0e_1)_{ij} dS_j + \lim_{L\to\infty} V_i \int_{S_L} ({}_{-1}p_1)_{ij} dS_j,\end{aligned}$$

where $\boldsymbol{\epsilon}$ is a constant vector and $({}_0e_1)_{ij}$ is the rate of strain tensor of the velocity homogeneous in r^0 in the asymptotic expansion of $\mathbf{u}_1$. It has been noted that the corresponding pressure is zero from equation (6.12*b*). Now

$$({}_0\mathbf{u}_1) = \tfrac{3}{16}(\boldsymbol{\alpha}.\nabla)[r(-3\mathbf{f}_0 + \mathbf{r}.\mathbf{f}_0\mathbf{r}/r^2)] + \tfrac{3}{16}(3\mathbf{f}_0 - \boldsymbol{\alpha}.\mathbf{f}_0\boldsymbol{\alpha}).$$

Hence the value of $({}_0e_1)_{ij}$ is due only to the term

$$\tfrac{3}{16}(\boldsymbol{\alpha}.\nabla)[r(-3\mathbf{f}_0 + \mathbf{r}.\mathbf{f}_0\mathbf{r}/r^2)]$$

since $\tfrac{3}{16}(3\mathbf{f}_0 - \boldsymbol{\alpha}.\mathbf{f}_0\boldsymbol{\alpha})$ is a constant vector. Therefore $({}_0e_1)_{ij}$ remains unaltered, whereas $\{V_i + [\mathbf{s}(\boldsymbol{\epsilon})]_i\} ({}_0e_1)_{ij} dS_j$ changes sign on replacing $\mathbf{r}$ by $-\mathbf{r}$. Therefore

$$\lim_{L\to\infty} \int_{S_L} \{V_i + [\mathbf{s}(\boldsymbol{\epsilon})]_i\} ({}_0e_1)_{ij} dS_j = 0,$$

whence
$$(I_1) = \lim_{L\to\infty} V_i \int_{S_L} ({}_{-1}p_1)_{ij} dS_j.$$

Now $\int_{S_L} ({}_0p_1)_{ij} dS_j = 0$ as may be shown by replacing $-\mathbf{r}$ by $\mathbf{r}$. Hence

$$(I_1) = V_i \lim_{L\to\infty} \int_{S_L} (p_1)_{ij} dS_j.$$

Thus for equation (6.14) we obtain

$$(I_1) = 6\pi V_i (f_1)_i + \lim_{L\to\infty} V_i \int_{S_L} (u_0)_i (u_0)_j dS_j.$$

But, from (7.14), the last integral is $-(I_4)$; hence,

$$\{(I_1)+(I_4)\} = 6\pi V_i(f_1)_i. \tag{7.15}$$

Next, consider (I_2). We note that

$$\mathbf{u}^* = \mathbf{V}+\frac{1}{r}(\boldsymbol{\epsilon}+\mathbf{r}\,.\,\boldsymbol{\epsilon}\mathbf{r}/r^2)+O(r^{-2}),$$

the corresponding pressure being $2(\boldsymbol{\epsilon}\,.\,\mathbf{r})/r^3+O(r^{-3})$. Since

$$p^*_{ij} = -p^*\delta_{ij}+\{(u^*)_{i,j}+(u^*)_{j,i}\},$$

it follows that p^*_{ij} is $O(r^{-2})$ and the term $({}_{-1}p^*)_{ij}$ changes sign on replacing $\mathbf{r}$ by $-\mathbf{r}$.

Hence contributions to (I_2) come only from ${}_0\mathbf{u}_1$. It follows that

$$(I_2) = \lim_{L\to\infty}\int_{S_L}\tfrac{3}{16}(3\mathbf{f}_0-\boldsymbol{\alpha}\,.\,\mathbf{f}_0\boldsymbol{\alpha})_i\,(p^*)_{ij}\,dS_j, \tag{7.16}$$

where it has been noted that the other term

$$\tfrac{3}{16}(\boldsymbol{\alpha}\,.\,\nabla)\,[r(-3\mathbf{f}_0+\mathbf{r}\,.\,\mathbf{f}_0\mathbf{r}/r^2)],$$

in ${}_0\mathbf{u}_1$ gives no contribution to (I_2) (as may be seen by replacing $\mathbf{r}$ by $-\mathbf{r}$). From equation (7.2), integrating over the volume V_L and letting $L\to\infty$, we obtain

$$\int_B (p^*)_{ij}\,dS_j+\lim_{L\to\infty}\int_{S_L}(p^*)_{ij}\,dS_j = 0.$$

Hence
$$(f^*)_i = \lim_{L\to\infty}\frac{1}{6\pi}\int_{S_L}(p^*)_{ij}\,dS_j, \tag{7.17}$$

where $(f^*)_i$ is the dimensionless Stokes force due to $(\mathbf{u}^*, p^*)$. From equations (7.16) and (7.17)

$$(I_2) = 6\pi\tfrac{3}{16}\{3(f_0)_i-\alpha_j(f_0)_j\,\alpha_i\}\,(f^*)_i. \tag{7.18}$$

For Stokes flow it has been shown (Brenner 1963) that the (dimensional) Stokes force on a body of arbitrary shape, past which fluid streams with (dimensional) velocity $\mathbf{W}'$, can be expressed in the invariant form $6\pi\mu c\phi_{ij}(\mathbf{W}')_j$ where ϕ_{ij} is a *symmetric* tensor, termed the (dimensionless) Stokes resistance tensor. (This tensor differs by a factor of 6π from the one used in a previous paper by Brenner 1961.) Hence the dimensionless force due to a dimensionless velocity $\mathbf{W}$ is $\phi_{ij}W_j$. Consequently

$$(f_0)_i = \phi_{ik}\alpha_k, \quad (f^*)_i = \phi_{ik}V_k. \tag{7.19}$$

Therefore
$$(I_2) = \tfrac{3}{16}(6\pi)\,(3\phi_{im}\alpha_m-\phi_{jl}\alpha_l\alpha_j\alpha_i)\,\phi_{ik}V_k. \tag{7.20}$$

Consider now (I_3). When $\mathbf{r}$ is replaced by $-\mathbf{r}$, the terms homogeneous in r^0 and r^{-1} in the asymptotic expansions of both $\mathbf{u}_0$ and $\mathbf{u}^*$ remain unaltered, whereas $d\mathbf{S}$ changes sign. Hence the only contributions to (I_3) can come from cases where one of the three factors in the integrand is homogeneous in r^{-2} while the rest is homogeneous in r^0. Thus

$$(I_3) = \int_{S_L}\{\alpha_i\alpha_j({}_{-2}u^*)_i+\alpha_iV_i({}_{-2}u_0)_j+\alpha_jV_i({}_{-2}u_0)_i\}\,dS_j.$$

Any sphere S_L will do in evaluating this integral since the integrands are all homogeneous in r^{-2}, whereas dS_j is homogeneous in r^2.

But, as discussed in §4,

$$\int_{S_L} ({}_{-2}u_0)_j\, dS_j = 0,$$

whence
$$(I_3) = \int_{S_L} \{\alpha_i \alpha_j ({}_{-2}u^*)_i + \alpha_j V_i ({}_{-2}u_0)_i\}\, dS_j. \tag{7.21}$$

Lastly, consider (I_4). By arguments similar to those used for (I_3) we obtain

$$(I_4) = -\int_{S_L} V_i \alpha_j ({}_{-2}u_0)_i\, dS_j. \tag{7.22}$$

Substitution of (7.15) and (7.20) to (7.22) into (7.10) yields

$$6\pi[(f_1)_i - \tfrac{3}{16}\{3\phi_{km}\phi_{ki} - \alpha_j \phi_{jl}\alpha_l \phi_{mi}\}\alpha_m] V_i = \alpha_i \alpha_j (I_5)_{ij} - (I_6), \tag{7.23}$$

in which
$$(I_5)_{ij} = \int_{S_L} ({}_{-2}u^*)_i\, dS_j, \quad (I_6) = \int_{\Gamma} (u_0)_i (e^*)_{ij} (u_0)_j\, dV. \tag{7.24}$$

Because of the linearity of the Stokes equations and the boundary conditions, there exists a tensor $K_{ij}(\mathbf{r})$ (not, in general, symmetric) which is a function of position such that
$$(u_0)_i = K_{ij}(\mathbf{r})\, \alpha_j, \quad (u^*)_i = K_{ij}(\mathbf{r})\, V_j. \tag{7.25}$$

This tensor can be determined from a knowledge of the Stokes velocity fields arising from streaming flows parallel to *any* three non-coplanar directions. Define $({}_tK_{ij})$ to be that part of K_{ij} which is homogeneous in r^t in its asymptotic expansion for large values of r. Then for $(I_5)_{ij}$ we obtain

$$(I_5)_{ij} = V_k \int_{S_L} ({}_{-2}K_{ik})\, dS_j. \tag{7.26}$$

Consider (I_6). It is easily shown that

$$(e^*)_{ij} = \tfrac{1}{2}(K_{il,j} + K_{jl,i})(V_l - V_k \alpha_k \alpha_l) + V_k \alpha_k (e_0)_{ij}.$$

Thus,

$$(I_6) = V_k \alpha_k (I_7) + (V_l - V_k \alpha_k \alpha_l)\alpha_m \alpha_n \int_{\Gamma} K_{im} K_{jn} \tfrac{1}{2}(K_{il,j} + K_{jl,i})\, dV, \tag{7.27}$$

in which
$$\begin{aligned}(I_7) &= \int_{\Gamma} (u_0)_i (e_0)_{ij} (u_0)_j\, dV = \int_{\Gamma} (u_0)_i (u_0)_j (u_0)_{i,j}\, dV \\ &= \int_{\Gamma} \{\tfrac{1}{2}u_0^2 (u_0)_j\}_{,j}\, dV \quad \text{since} \quad \nabla . \mathbf{u}_0 = 0 \\ &= \lim_{L\to\infty} \int_{S_L} \tfrac{1}{2}u_0^2 (u_0)_j\, dS_j \quad \text{since} \quad \mathbf{u}_0 = 0 \quad \text{on} \quad B.\end{aligned}$$

By using arguments similar to those employed in the evaluation of (I_3), we find that the last integral reduces to

$$(I_7) = \frac{1}{2}\int_{S_L} \{\alpha_i \alpha_i ({}_{-2}u_0)_j + 2\alpha_i \alpha_j ({}_{-2}u_0)_i\}\, dS_j.$$

But,
$$\int_{S_L} ({}_{-2}u_0)_j\, dS_j = 0;$$

hence
$$(I_7) = \alpha_i \alpha_j \int_{S_L} ({}_{-2}u_0)_i\, dS_j = \alpha_i \alpha_j \alpha_l \int_{S_L} ({}_{-2}K_{il})\, dS_j. \tag{7.28}$$

Substituting into (7.27), we obtain (I_6).

The latter expression, along with (7.26), when substituted in (7.23) gives, on changing indices,

$$6\pi V_i[(f_1)_i - \tfrac{3}{16}\{3\phi_{ij} - \delta_{ij}(\alpha_k\phi_{kl}\alpha_l)\}\phi_{jm}\alpha_m]$$
$$= V_i\left[\delta_{il}\alpha_m\alpha_j\int_{S_L}(_{-2}K_{ml})\,dS_j - \alpha_m\alpha_j\alpha_i\alpha_l\int_{S_L}(_{-2}K_{ml})\,dS_j - \alpha_k\alpha_l(\delta_{ip}-\alpha_i\alpha_p)\int_\Gamma K_{mk}K_{jl}\tfrac{1}{2}(K_{mp,j}+K_{jp,m})\,dV\right]. \quad (7.29)$$

The vector $\mathbf{V}$ was chosen arbitrarily and so

$$(f_1)_i = \tfrac{3}{16}\{3\phi_{ij} - \delta_{ij}(\alpha_k\phi_{kl}\alpha_l)\}\phi_{jm}\alpha_m + \frac{1}{6\pi}\alpha_m\alpha_j(\delta_{il}-\alpha_i\alpha_l)B_{lmj}, \quad (7.30)$$

in which B_{lmj} is a (dimensionless) third-order tensor given by

$$B_{lmj} = \int_{S_L}(_{-2}K_{ml})\,dS_j - \int_\Gamma K_{km}K_{pj}\tfrac{1}{2}(K_{kl,p}+K_{pl,k})\,dV. \quad (7.31)$$

Now define $(v_i)_j$ as the jth component of the velocity field due to a Stokes flow around the body due to a uniform stream of unit strength in the ith direction. Similarly, define $(e_i)_{jk}$ as the (j,k) component of the rate-of-strain tensor corresponding to this flow. Therefore

$$K_{ij} = (v_j)_i, \quad (7.32)$$

as can be seen from equations (7.25). Hence from equation (7.31)

$$B_{lmj} = \int_{S_L}(_{-2}v_l)_m\,dS_j - \int_\Gamma (v_m)_k(e_l)_{kp}(v_j)_p\,dV. \quad (7.33)$$

Also, if A_{lmj} is defined as $$A_{lmj} = \tfrac{1}{2}(B_{lmj}+B_{ljm}), \quad (7.34)$$

we note that B_{lmj} in equation (7.30) may be replaced by A_{lmj}. From equation (7.19)

$$(f_0)_i = \delta_{ij}\phi_{jm}\alpha_m. \quad (7.35)$$

Putting $\alpha_i = U_i/U$ in equations (7.30) and (7.35), and substituting into equation (4.8) for $\mathbf{f}$, we obtain

$$(f)_i = [\delta_{ij} + \tfrac{3}{16}R\{3\phi_{ij} - \delta_{ij}(U_k\phi_{kl}U_l/U^2)\}]\phi_{jm}(U_m/U) + \frac{1}{6\pi}R[(U_mU_j/U^2)\{\delta_{il} - (U_iU_l/U^2)\}]A_{lmj} + o(R). \quad (7.36)$$

Upon expressing this equation in dimensional variables, we obtain the main results of our present section, namely, that the force $\mathbf{F}$ on the particle to $O(R)$, due to a uniform velocity $\mathbf{U}$ at infinity, is given by

$$F_i = (^1F)_i + (^2F)_i + \{\mu c U\}o(R), \quad (7.37a)$$

where

$$\left.\begin{aligned}(^1F)_i &= 6\pi\mu cU[\delta_{ij} + \tfrac{3}{16}R\{3\phi_{ij} - \delta_{ij}(U_k\phi_{kl}U_l/U^2)\}]\phi_{jm}(U_m/U),\\ (^2F)_i &= \mu cUR[(U_mU_j/U^2)\{\delta_{il} - (U_iU_l/U^2)\}]A_{lmj},\\ A_{lmj} &= \frac{1}{2}\left\{\int_{S_L}(_{-2}v_l)_m\,dS_j + \int_{S_L}(_{-2}v_l)_j\,dS_m\right\} - \int_\Gamma (v_m)_k(e_l)_{kp}(v_j)_p\,dV.\end{aligned}\right\} \quad (7.37b)$$

A_{lmj} is symmetric in the indices m and j, and is determined by the Stokes velocity fields due to flows at infinity along each of the three axes. We have used the symbols $({}^1F)_i$ and $({}^2F)_i$ to mean those parts of the force F_i which are respectively reversed and unaltered by a reversal of flow at infinity (to the order in R considered). This may be checked by replacing U_i by $-U_i$.

We notice that for ${}^2\mathbf{F}$ given by equations (7.37)

$$U_i({}^2F)_i = 0. \tag{7.38}$$

Thus ${}^2\mathbf{F}$ is simply a lift force, being perpendicular to the stream velocity $\mathbf{U}$ at infinity. ${}^2\mathbf{F}$ does not, however, constitute the entire lift force experienced by the body; ${}^1\mathbf{F}$ also contains a component at right-angles to $\mathbf{U}$.

In deriving equations (7.37) we have at no point stated specifically what value of c is to be taken, and so they must be independent of c. That this is indeed so is easily seen by noting that ϕ_{ij} and A_{lmj} are proportional to c^{-1} and c^{-2}, respectively.

8. Discussion of the force to O(R)

To the first-order in R, the vector force $\mathbf{F}$ on a body of arbitrary shape is given in terms of the stream velocity $\mathbf{U}$ at infinity by equation (7.37).

Since ${}^2\mathbf{F}$ is a lift force, at right-angles to $\mathbf{U}$, it follows that the *drag* (i.e. the component of the vector force parallel to $\mathbf{U}$) experienced by an arbitrary particle is given by that component of ${}^1\mathbf{F}$ lying parallel to $\mathbf{U}$. Thus if the 1-axis, say, corresponds to the direction of streaming flow, then the drag may be calculated from the expression

$$F_i = 6\pi\mu c U[\delta_{ij} + \tfrac{3}{16}R\{3\phi_{ij} - \delta_{ij}(U_k\phi_{kl}U_l/U^2)\}]\,\phi_{jm}(U_m/U) + \{\mu c U\}\,o(R) \tag{8.1}$$

by setting $i = 1$ and $U_j = \delta_{1j}U$. In § 10 we generalize and elaborate on this drag formula more fully. Observe that the 'reversibility' of ${}^1\mathbf{F}$ implies that the magnitude of this drag remains invariant to a reversal of the flow at infinity, at least to the order in R considered. This seems a rather surprising result, for in the absence of symmetry the flow field itself is not reversed to this order.

More general results, pertaining to the *vector* force on the body, may be deduced for bodies possessing certain symmetries. In particular, if the symmetry of the body demands that the vector force be wholly reversed in reverse flow (to all orders in R), irrespective of the direction of $\mathbf{U}$, then the 'irreversibility' of ${}^2\mathbf{F}$ requires that the latter vanish. Hence from (7.37), the vector force on the body (e.g. an ellipsoid in arbitrary orientation) is given by (8.1) for all directions of flow. This formula is identical to that of Brenner (1961) and Chester (1962). In a wider sense, if symmetry requires only that a certain component of $\mathbf{F}$ be reversed in reverse flow, then this component of ${}^2\mathbf{F}$ must be zero and the same component of $\mathbf{F}$ is given correctly by (8.1)—in agreement with Chester's (1962) conclusion.

We now list some types of symmetry‡ for bodies for which one may conclude that some or all of the components of force on the body to $O(R)$ are given by equation (8.1). A detailed proof is given in Appendix 1.

‡ For some further comments, see end of paper.

Consider an arbitrary set of mutually perpendicular axes labelled 1, 2 and 3. Let the plane 1 contain both axes 2 and 3. Then we have the following cases.

(I) $({}^2F)_i$ *vanishes for all directions of* **U** if the body transforms into itself under

Either (i) Triple reflexion (8.2*a*)

(i.e. a successive reflexion in each of the planes 1, 2 and 3)

Or (ii) Rotations through $\frac{1}{2}\pi$ about each of axes 1, 2 and 3. (8.2*b*)

Neither of these conditions is included in the other. Also it follows from symmetry considerations that a body satisfying (i) is one for which the *vector* force is reversed to all orders in R if the velocity at infinity is reversed. Hence $({}^2F)_i$ must be zero, and in fact this is a case where Chester's (1962) result may be used.

(II) $({}^2F)_i$ *vanishes for* **U** *lying in the plane* 3 if the body transforms into itself under

Rotation through π about axis 1; Rotation through $\frac{1}{2}\pi$ about axis 3. (8.3)

(III) $({}^2F)_3$ *vanishes for* **U** *lying in the plane* 3 if the body transforms into itself under

Reflexion in plane 3. (8.4)

Here, symmetry considerations dictate that the component of lift on the body in direction 3 be zero to all orders in R. Hence $({}^2F)_3$ and $({}^1F)_3$ must each be zero.

(IV) $({}^2F)_1$ *and* $({}^2F)_2$ *both vanish for* **U** *lying in the plane* 3 if the body transforms into itself under

Rotation through π about axis 3. (8.5)

For bodies satisfying this condition, symmetry requires that a reversal of flow at infinity reverse that component of **F** lying in plane 3, to all orders in R. Hence $({}^2F)_i$ must be zero. (This is another case where Chester's result would apply.)

(V) $({}^2F)_i$ *vanishes for* **U** *lying along the axis* 1 if the body transforms into itself under

Rotation through π about axis 1. (8.6)

It is seen, by symmetry, that such a body is one for which there can be no lift to all orders in the Reynolds number. Hence $({}^1F)_2$, $({}^1F)_3$, $({}^2F)_2$ and $({}^2F)_3$ are all zero. Also $({}^2F)_1$ is zero since $({}^2F)_i U_i$ is always zero.

(VI) $({}^2F)_1$ *and* $({}^2F)_3$ *both vanish for* **U** *lying along the axis* 1 if the body transforms into itself under

Reflexion in plane 2. (8.7)

This result is a combination of the case (III), together with the result of the vanishing of the contribution of $({}^2F)_i$ to the drag.

By considering uniform streaming past a slightly deformed sphere, it may be shown that the contribution ${}^2\mathbf{F}$ to the vector force on the body is not always zero. A detailed proof is given in Appendix 2.

9. Oseen force

In the case of the sphere (Proudman & Pearson 1957) and spheroid (Breach 1961), it has been pointed out that to $O(R)$ the drag determined by the singular perturbation technique is identical to that predicted by the classical Oseen equations, despite the fact that the latter do not furnish the correct asymptotic behaviour of the Navier-Stokes equations to the first order in R. In order to establish the general circumstances for which this conclusion is valid, we now proceed to derive, on the basis of the classical Oseen equations, the counterpart of (7.37) for the Oseen force.

Thus we wish to solve the equations

$$\mu\nabla'.\nabla'\mathbf{u}-\nabla'p' = \rho\mathbf{U}.\nabla'\mathbf{u}', \quad \nabla'.\mathbf{u}' = 0, \tag{9.1a,b}$$

together with the boundary conditions

$$\mathbf{u}' = 0 \quad \text{on} \quad B', \tag{9.2}$$

$$\mathbf{u}' \to U\boldsymbol{\alpha} \quad \text{as} \quad r' \to \infty, \tag{9.3}$$

where all the variables are dimensional (cf. equations (2.1) to (2.3)).

Define dimensionless variables as in § 2 to give

$$\nabla^2\mathbf{u}-\nabla p = R\boldsymbol{\alpha}.\nabla\mathbf{u}, \quad \nabla.\mathbf{u} = 0, \tag{9.4}$$

$$\mathbf{u} = 0 \quad \text{on} \quad B, \tag{9.5}$$

$$\mathbf{u} \to \boldsymbol{\alpha} \quad \text{as} \quad r \to \infty. \tag{9.6}$$

This system of equations may be solved to $O(R)$ by forming inner and outer expansions (§ 3) comparable to those employed in solving the analogous equations (2.6) to (2.8).

The zeroth-order inner approximation (§ 4) and first-order outer approximation (§ 5) are unaltered. However, for the first-order inner approximation, $(\mathbf{u}_1, p_1)$ now satisfies the equations

$$\nabla^2\mathbf{u}_1-\nabla p_1 = \boldsymbol{\alpha}.\nabla\mathbf{u}_0, \quad \nabla.\mathbf{u}_1 = 0, \tag{9.7}$$

and

$$\left.\begin{aligned} &\mathbf{u}_1 = 0 \quad \text{on} \quad B\\ &\mathbf{u}_1 \sim \tfrac{3}{16}\{(\boldsymbol{\alpha}.\nabla)[r(-3\mathbf{f}_0+\mathbf{r}.\mathbf{f}_0\mathbf{r}/r^2)]+(3\mathbf{f}_0-\boldsymbol{\alpha}.\mathbf{f}_0\boldsymbol{\alpha})\},\\ &p_1 \sim o(r^{-1}) \quad \text{as} \quad r\to\infty,\end{aligned}\right\} \tag{9.8}$$

where

$$\boldsymbol{\alpha}.\nabla\mathbf{u}_0 = -\tfrac{3}{4}\boldsymbol{\alpha}.\nabla[\mathbf{s}(\mathbf{f}_0)]+\lambda_{pq}(\boldsymbol{\alpha}.\nabla)(\mathbf{i}_p.\nabla)[\mathbf{s}(\mathbf{i}_q)]+O(r^{-4}). \tag{9.9}$$

By the methods used in § 6, we find that the values of $\mathbf{u}_1$ and p_1 are now

$$\begin{aligned}\mathbf{u}_1 = &+\tfrac{3}{16}(\boldsymbol{\alpha}.\nabla)[r(-3\mathbf{f}_0+\mathbf{r}.\mathbf{f}_0\mathbf{r}/r^2)]+\tfrac{3}{16}(3\mathbf{f}_0-\boldsymbol{\alpha}.\mathbf{f}_0\boldsymbol{\alpha})\\ &-\tfrac{1}{4}\lambda_{pq}(\boldsymbol{\alpha}.\nabla)(\mathbf{i}_p.\nabla)[r(-3\mathbf{i}_q+\mathbf{r}.\mathbf{i}_q\mathbf{r}/r^2)]+[\mathbf{s}(\boldsymbol{\delta}'')]+O(r^{-2}),\end{aligned} \tag{9.10a}$$

$$p_1 = [t(\boldsymbol{\delta}'')]+O(r^{-3}), \tag{9.10b}$$

where the constant vector $\boldsymbol{\delta}''$ is determined by the boundary condition that $\mathbf{u}_1 = 0$ on the surface B. Comparing equations (9.10) with (6.12), we see that the values of $\mathbf{u}_1$ and p_1 are not identical. Thus we have verified that the classical Oseen equations do not give the correct asymptotic behaviour to $O(R)$.

The dimensionless force $\mathbf{f}_1$ (given before by (6.14)) is now

$$(f_1)_i = \frac{1}{6\pi}\int_{S_L}(p_1)_{ij}dS_j-\frac{1}{6\pi}\int_{S_L}(u_0)_i\alpha_j dS_j. \tag{9.11}$$

Continuing our analysis of the Oseen equations, as in § 7, we obtain the analogue of equation (7.9) as

$$\int_{B+S_L}\{(u^*)_i(p_1)_{ij}-(u_1)_i(p^*)_{ij}-(u_0)_i\alpha_j(u^*)_i\}dS_j+\int_{V_L}(u_0)_i\alpha_j(u^*)_{i,j}dV = 0. \tag{9.12}$$

This then leads to a relation identical to (7.23) except that we now have

$$(I_6) = \int_\Gamma (u_0)_i \, (u^*)_{i,j} \alpha_j dV.$$

This, in turn, leads to a relation of the same form as (7.30) except that now

$$B_{lmj} = \int_{S_L} (_{-2}v_l)_m dS_j - \int_\Gamma (v_m)_k \, (v_l)_{k,j} dV. \tag{9.13}$$

Hence the force, $\overline{\mathbf{F}}$ say, on the particle as calculated from Oseen's equation is

$$\overline{F}_i = ({}^1\overline{F})_i + ({}^2\overline{F})_i + \{\mu c U\} \, o(R), \tag{9.14a}$$

where

$$\left.\begin{aligned} ({}^1\overline{F})_i &= 6\pi\mu c U[\delta_{ij} + \tfrac{3}{16}R\{3\phi_{ij} - \delta_{ij}(U_k \phi_{kl} U_l/U^2)\}] \, \phi_{jm}(U_m/U), \\ ({}^2\overline{F})_i &= \mu c U R[(U_m U_j/U^2)\{\delta_{il} - (U_i U_l/U^2)\}] \, \overline{A}_{lmj}, \\ \text{and} \quad \overline{A}_{lmj} &= \frac{1}{2}\left\{\int_{S_L} (_{-2}v_l)_m \, dS_j + \int_{S_L} (_{-2}v_l)_j \, dS_m\right\} \\ &\quad -\frac{1}{2}\left\{\int_\Gamma (v_m)_k \, (v_l)_{k,j} dV + \int_\Gamma (v_j)_k \, (v_l)_{k,m} dV\right\}. \end{aligned}\right\} \tag{9.14b}$$

It should be noted that $({}^1\overline{F})_i$ is identical to $({}^1F)_i$ and that the formula for $({}^2\overline{F})_i$ is the same as that for $({}^2F)_i$ with A_{lmj} replaced by a different tensor $\overline{A}_{lmj}$. The latter, like A_{lmj}, is symmetric in the indices m and j. Thus $({}^2\overline{F})_i$, like $({}^2F)_i$, is a lift force. It therefore follows that the *drag* on a body of arbitrary shape, as determined by the classical Oseen method, is identical to that predicted by the singular perturbation technique, at least to $O(R)$.

It is clear that the discussion in §8 and Appendix 1, involving A_{lmj}, may equally well be applied to $\overline{A}_{lmj}$. Thus, if for some direction of flow, a body has one of the types of symmetry described in §8, causing a particular component of $({}^2F)_i$ to vanish, then the *same* component of $({}^2\overline{F})_i$ must vanish. Thus for *that particular component*

$$\overline{F}_i = ({}^1\overline{F})_i + \{\mu c U\} \, o(R).$$

Hence

$$\overline{F}_i = F_i + \{\mu c U\} \, o(R),$$

i.e. for that particular component, the value of the Oseen force on the body to $O(R)$ is identical to that predicted by the singular perturbation technique.

However, by considering again the deformed sphere (see the last paragraph of §8) it may be shown that ${}^2\overline{\mathbf{F}}$ is *not* always the same as ${}^2\mathbf{F}$. We conclude, therefore, that the Oseen method does not always furnish the correct *lift* component. A detailed proof of this statement is given in Appendix 2.

10. Second-order inner approximation

In this section we resume the singular perturbation analysis terminated in §7. Thereby, we obtain the next higher approximation, in orders of R, to the force on a body of arbitrary shape.

We now add to the inner expansions (3.1), (3.2),

$$\mathbf{u} = \mathbf{u}_0 + R\mathbf{u}_1 + o(R), \quad p = p_0 + Rp_1 + o(R),$$

the additional terms $\theta_2(R)\mathbf{u}_2$ and $\theta_2(R)p_2$, respectively. In the previous treatments of the sphere (Proudman & Pearson 1957) and spheroid (Breach 1961) it has been shown that the assumption $\theta_2(R) = R^2$ leads to a particular integral for $\mathbf{u}_2$ containing a term in $\log r$. This will be shown to be true in the general case, the term being of the form $\mathbf{d}\log r$, where $\mathbf{d}$ is a non-zero constant vector. When expressed in the outer variables,

$$R^2(\mathbf{d}\log r) = R^2(\mathbf{d}\log \tilde{r}) - (R^2\log R)\,\mathbf{d}.$$

Thus there is apparently a term in $R^2\log R$ and one in $R^3\log R$ in the outer expansion of the velocity and pressure fields. Such a field, $(\bar{\mathbf{U}}_2, P_2)$ say, would satisfy the Oseen equations and the boundary conditions

$$\bar{\mathbf{U}}_2 \to \mathbf{d} \quad \text{as} \quad \tilde{r} \to 0, \quad \bar{\mathbf{U}}_2 \to 0 \quad \text{as} \quad \tilde{r} \to \infty.$$

By considering $\boldsymbol{\omega} = \tilde{\nabla} \wedge \bar{\mathbf{U}}_2$, we find that $\boldsymbol{\omega}$ satisfies the differential equations

$$\tilde{\nabla}^2\boldsymbol{\omega} = \boldsymbol{\alpha}.\tilde{\nabla}\boldsymbol{\omega}, \quad \tilde{\nabla}.\boldsymbol{\omega} = 0.$$

Upon dot-multiplying the first of these equations by $\boldsymbol{\omega}$ and employing the Divergence theorem, we eventually obtain

$$\int_{\tilde{S}_L+\tilde{S}_l} [\omega_i\omega_{i,j} - \alpha_j(\tfrac{1}{2}\omega^2)\, d\tilde{S}_j = \int_{\tilde{V}} \omega_{i,j}\omega_{i,j}\, d\tilde{V},$$

where $\tilde{S}_L$ and $\tilde{S}_l$ are the surfaces of spheres of radii $\tilde{r} = L$ and l $(l \ll L)$ with centre at the origin. On letting $l \to 0$ and $L \to \infty$ and observing that the right-hand integral is non-negative, we may prove that $\boldsymbol{\omega} = 0$. Hence $\tilde{\nabla} \wedge \bar{\mathbf{U}}_2 = \tilde{\nabla}.\bar{\mathbf{U}}_2 = 0$. It may be shown that it is impossible to satisfy these equations for $\bar{\mathbf{U}}_2$ with the above boundary conditions. Hence there can be no $R^2\log R$ term in the outer expansion. There must therefore be a term in the inner expansion, which for large r is $+(R^2\log R)\,\mathbf{d}$, such that when added to the term $R^2(\mathbf{d}\log r)$ (arising from the particular integral of $(\mathbf{u}_2, p_2)$) gives rise only to a term $R^2(\mathbf{d}\log\tilde{r})$ when expressed in outer variables. Thus the correct form of the inner expansion is

$$\mathbf{u} = \mathbf{u}_0 + R\mathbf{u}_1 + (R^2\log R)\,\mathbf{u}_2 + R^2\bar{\mathbf{u}}_2 + o(R^2), \tag{10.1}$$

with a comparable expansion for p.

Upon substituting these into the Navier–Stokes equations (2.6) we find that $(\mathbf{u}_2, p_2)$ satisfies Stokes equations,

$$\left.\begin{aligned} &\nabla^2\mathbf{u}_2 - \nabla p_2 = 0, \quad \nabla.\mathbf{u}_2 = 0, \\ &\text{and the boundary conditions} \\ &\mathbf{u}_2 = 0 \quad \text{on} \quad B, \\ &\mathbf{u}_2 \to \mathbf{d} \quad \text{as} \quad r \to \infty, \end{aligned}\right\} \tag{10.2}$$

and $(\bar{\mathbf{u}}_2, \bar{p}_2)$ satisfies

$$\nabla^2\bar{\mathbf{u}}_2 - \nabla\bar{p}_2 = \mathbf{u}_0.\nabla\mathbf{u}_1 + \mathbf{u}_1.\nabla\mathbf{u}_0, \quad \nabla.\bar{\mathbf{u}}_2 = 0. \tag{10.3}$$

The $\log r$ term which eventually arises in $\bar{\mathbf{u}}_2$ can only be derived from those terms in $(\mathbf{u}_0.\nabla\mathbf{u}_1 + \mathbf{u}_1.\nabla\mathbf{u}_0)$ which are $O(r^{-2})$. Thus, since $\mathbf{u}_0$ and $\mathbf{u}_1$ are both of $O(r^0)$, each need only be known correctly to $O(r^{-1})$. The asymptotic forms of these fields, given in (4.11) and (6.12), are accurate to this degree of approximation. Now

$$\begin{aligned}\mathbf{u}_0.\nabla\mathbf{u}_1+\mathbf{u}_1.\nabla\mathbf{u}_0 = {} & \tfrac{3}{16}(\boldsymbol{\alpha}.\nabla)(\boldsymbol{\alpha}.\nabla)[r(-3\mathbf{f}_0+\mathbf{r}.\mathbf{f}_0\mathbf{r}/r^2)] \\ & +(\boldsymbol{\alpha}.\nabla)[-\tfrac{1}{4}\lambda_{pq}(\boldsymbol{\alpha}.\nabla)(\mathbf{i}_p.\nabla)\{r(-3\mathbf{i}_q+\mathbf{r}.\mathbf{i}_q\mathbf{r}/r^2)\} \\ & +\tfrac{9}{32}r^{-2}\{\mathbf{f}_0.\mathbf{r}\mathbf{f}_0-f_0^2\mathbf{r}+2(\mathbf{f}_0.\mathbf{r})^2\mathbf{r}/r^2\}+[\mathbf{s}(\boldsymbol{\delta})]] \\ & -\tfrac{3}{4}[\mathbf{s}(\mathbf{f}_0)].\nabla\{\tfrac{3}{16}(\boldsymbol{\alpha}.\nabla)[r(-3\mathbf{f}_0+\mathbf{r}.\mathbf{f}_0\mathbf{r}/r^2)]\} \\ & +\tfrac{3}{16}(3\mathbf{f}_0-\boldsymbol{\alpha}.\mathbf{f}_0\boldsymbol{\alpha}).\nabla\{-\tfrac{3}{4}[\mathbf{s}(\mathbf{f}_0)]\} \\ & +\tfrac{3}{16}\{(\boldsymbol{\alpha}.\nabla)[r(-3\mathbf{f}_0+\mathbf{r}.\mathbf{f}_0\mathbf{r}/r^2)]\}.\nabla\{-\tfrac{3}{4}[\mathbf{s}(\mathbf{f}_0)]\} \\ & +O(r^{-3}).\end{aligned} \tag{10.4}$$

Consider the equations

$$\nabla^2\mathbf{u}^{(\mathrm{iv})}-\nabla p^{(\mathrm{iv})} = r(-3\mathbf{i}_q+\mathbf{r}.\mathbf{i}_q\mathbf{r}/r^2), \quad \nabla.\mathbf{u}^{(\mathrm{iv})}=0. \tag{10.5}$$

By taking spherical polar axes, with axis in the direction of $\mathbf{i}_q$ and solving in terms of a stream function we find that

$$\mathbf{u}^{(\mathrm{iv})} = \tfrac{1}{18}r^3[-5\mathbf{i}_q+3\mathbf{r}.\mathbf{i}_q\mathbf{r}/r^2], \quad p^{(\mathrm{iv})}=0. \tag{10.6}$$

By the arguments used in § 6 we see that the solution of

$$\begin{aligned}\nabla^2\mathbf{u}-\nabla p = {} & \tfrac{3}{16}(\boldsymbol{\alpha}.\nabla)(\boldsymbol{\alpha}.\nabla)[r(-3\mathbf{f}_0+\mathbf{r}.\mathbf{f}_0\mathbf{r}/r^2)] \\ & +(\boldsymbol{\alpha}.\nabla)[-\tfrac{1}{4}\lambda_{pq}(\boldsymbol{\alpha}.\nabla)(\mathbf{i}_p.\nabla)\{r(-3\mathbf{i}_q+\mathbf{r}.\mathbf{i}_q\mathbf{r}/r^2)\},\end{aligned}$$

with $\nabla.\mathbf{u}=0$, is $p=0$

and
$$\begin{aligned}\mathbf{u} = {} & \tfrac{3}{16}(\boldsymbol{\alpha}.\nabla)(\boldsymbol{\alpha}.\nabla)\{\tfrac{1}{18}r^3(-5\mathbf{f}_0+3\mathbf{r}.\mathbf{f}_0\mathbf{r}/r^2)\} \\ & +(\boldsymbol{\alpha}.\nabla)[-\tfrac{1}{4}\lambda_{pq}(\boldsymbol{\alpha}.\nabla)(\mathbf{i}_p.\nabla)\{\tfrac{1}{18}r^3(-5\mathbf{i}_q+3\mathbf{r}.\mathbf{i}_q\mathbf{r}/r^2)\}].\end{aligned}$$

Clearly this does not involve a $\log r$ term since we are differentiating a polynomial in r. Similarly, the solution of

$$\nabla^2\mathbf{u}-\nabla p = \boldsymbol{\alpha}.\nabla[\mathbf{s}(\boldsymbol{\delta})]+\tfrac{3}{16}(3\mathbf{f}_0-\boldsymbol{\alpha}.\mathbf{f}_0\boldsymbol{\alpha}).\nabla\{-\tfrac{3}{4}[\mathbf{s}(\mathbf{f}_0)]\},$$

with $\nabla.\mathbf{u}=0$, for $\mathbf{u}$ cannot involve $\log r$. We have here used the fact that the solution (6.5) of the equation (6.4) does not involve a $\log r$ term.

Hence we see that the part of the particular integral of equation (10.3) for $\bar{\mathbf{u}}_2$ which involves $\log r$ is the same as that part of the particular integral involving $\log r$ of the following equations:

$$\begin{aligned}\nabla^2\bar{\mathbf{u}}_2^*-\nabla\bar{p}_2^* = {} & \tfrac{9}{16}(\boldsymbol{\alpha}.\nabla)[r^{-2}\{\mathbf{f}_0.\mathbf{r}\mathbf{f}_0-f_0^2\mathbf{r}+2(\mathbf{f}_0.\mathbf{r})^2\mathbf{r}/r^2\}] \\ & -\tfrac{3}{4}[\mathbf{s}(\mathbf{f}_0)].\nabla\{\tfrac{3}{16}(\boldsymbol{\alpha}.\nabla)[r(-3\mathbf{f}_0+\mathbf{f}_0.\mathbf{r}\mathbf{r}/r^2)]\} \\ & +\tfrac{3}{16}\{(\boldsymbol{\alpha}.\nabla)[r(-3\mathbf{f}_0+\mathbf{f}_0.\mathbf{r}\mathbf{r}/r^2)]\}.\nabla\{-\tfrac{3}{4}[\mathbf{s}(\mathbf{f}_0)]\}+O(r^{-3}),\end{aligned} \tag{10.7a}$$

$$\nabla.\bar{\mathbf{u}}_2^* = 0. \tag{10.7b}$$

After a tedious but straightforward calculation (10.7*a*) yields

$$\begin{aligned}\nabla^2\bar{\mathbf{u}}_2^*-\nabla\bar{p}_2^* = {} & \tfrac{9}{64}[\{5\mathbf{f}_0.\boldsymbol{\alpha}/r^2-6\boldsymbol{\alpha}.\mathbf{r}\mathbf{f}_0.\mathbf{r}/r^4\}\mathbf{f}_0 \\ & +\{8f_0^2\boldsymbol{\alpha}.\mathbf{r}/r^4-26(\mathbf{f}_0.\mathbf{r})^2(\boldsymbol{\alpha}.\mathbf{r})/r^6+10\boldsymbol{\alpha}.\mathbf{f}_0\mathbf{r}.\mathbf{f}_0/r^4\}\mathbf{r} \\ & +\{4(\mathbf{f}_0.\mathbf{r})^2/r^4-3f_0^2/r^2\}\boldsymbol{\alpha}]+O(r^{-3}).\end{aligned} \tag{10.8}$$

Upon taking the divergence of this equation we find that

$$\nabla^2 \bar{p}_2^* = \tfrac{9}{8}[\mathbf{f}_0 . \boldsymbol{\alpha}\mathbf{r} . \mathbf{f}_0/r^4 - (\mathbf{f}_0 . \mathbf{r})^2 (\boldsymbol{\alpha} . \mathbf{r})/r^6] + O(r^{-4}). \tag{10.9}$$

A particular integral of this is

$$\bar{p}_2^* = \tfrac{3}{32}[(\mathbf{f}_0 . \mathbf{r})^2 (\boldsymbol{\alpha} . \mathbf{r})/r^4 + f_0^2 (\boldsymbol{\alpha} . \mathbf{r})/r^2 - 4(\mathbf{f}_0 . \boldsymbol{\alpha}) (\mathbf{r} . \mathbf{f}_0)/r^2] + O(r^{-2}). \tag{10.10}$$

This makes

$$\begin{aligned} \nabla \bar{p}_2^* = \tfrac{3}{32}[&\{-4\mathbf{f}_0 . \boldsymbol{\alpha}/r^2 + 2\boldsymbol{\alpha} . \mathbf{r}\mathbf{f}_0 . \mathbf{r}/r^4\}\mathbf{f}_0 \\ &+ \{-2f_0^2 \boldsymbol{\alpha} . \mathbf{r}/r^4 - 4(\mathbf{f}_0 . \mathbf{r})^2 (\boldsymbol{\alpha} . \mathbf{r})/r^6 + 8\boldsymbol{\alpha} . \mathbf{f}_0 \mathbf{r} . \mathbf{f}_0/r^4\}\mathbf{r} \\ &+ \{(\mathbf{f}_0 . \mathbf{r})^2/r^4 + f_0^2/r^2\}\boldsymbol{\alpha}] + O(r^{-3}). \end{aligned} \tag{10.11}$$

From equations (10.8) and (10.11) we obtain the following equations for $\bar{\mathbf{u}}_2^*$,

$$\begin{aligned} \nabla^2 \bar{\mathbf{u}}_2^* = \tfrac{3}{64}[&\{7\mathbf{f}_0 . \boldsymbol{\alpha}/r^2 - 14\boldsymbol{\alpha} . \mathbf{r}\mathbf{f}_0 . \mathbf{r}/r^4\}\mathbf{f}_0 \\ &+ \{20f_0^2 \boldsymbol{\alpha} . \mathbf{r}/r^4 - 86(\mathbf{f}_0 . \mathbf{r})^2 (\boldsymbol{\alpha} . \mathbf{r})/r^6 + 46\boldsymbol{\alpha} . \mathbf{f}_0 \mathbf{r} . \mathbf{f}_0/r^4\}\mathbf{r} \\ &+ \{14(\mathbf{f}_0 . \mathbf{r})^2/r^4 - 7f_0^2/r^2\}\boldsymbol{\alpha}] + O(r^{-3}), \end{aligned} \tag{10.12a}$$

$$\nabla . \bar{\mathbf{u}}_2^* = 0. \tag{10.12b}$$

A particular integral of (10.12*a*) is

$$\begin{aligned} \bar{\mathbf{u}}_2^* = \tfrac{3}{640}[&\{62(\mathbf{f}_0 . \boldsymbol{\alpha}) \log r + 52\boldsymbol{\alpha} . \mathbf{r}\mathbf{f}_0 . \mathbf{r}/r^2\}\mathbf{f}_0 \\ &+ \{-19f_0^2 \boldsymbol{\alpha} . \mathbf{r}/r^2 + 43(\mathbf{f}_0 . \mathbf{r})^2 \boldsymbol{\alpha} . \mathbf{r}/r^4 - 48\boldsymbol{\alpha} . \mathbf{f}_0 \mathbf{r} . \mathbf{f}_0/r^2\}\mathbf{r} \\ &+ \{-14f_0^2 \log r - 9(\mathbf{f}_0 . \mathbf{r})^2/r^2\}\boldsymbol{\alpha}] + O(r^{-1}). \end{aligned}$$

By direct differentiation it may be verified that this satisfies the continuity equation (10.12*b*).

The complementary function of the equations (10.3) is a solution of Stokes equations and so cannot contain a term in log r.

Thus, we have found that

$$\bar{\mathbf{u}}_2 = \tfrac{3}{320}\{31\,\mathbf{f}_0 . \boldsymbol{\alpha}\mathbf{f}_0 - 7f_0^2 \boldsymbol{\alpha}\}\log r + \begin{Bmatrix} \text{No other functions of } O(1) \\ \text{or } O(r) \text{ which contain } \log r \end{Bmatrix}. \tag{10.13}$$

Therefore, in equation (10.2),

$$\mathbf{d} = \tfrac{3}{320}\{31\mathbf{f}_0 . \boldsymbol{\alpha}\mathbf{f}_0 - 7f_0^2 \boldsymbol{\alpha}\}. \tag{10.14}$$

Hence, we see that $\mathbf{u}_2$ is just a Stokes flow around the body which tends to a uniform stream at infinity of velocity $\tfrac{3}{320}\{31\mathbf{f}_0 . \boldsymbol{\alpha}\mathbf{f}_0 - 7f_0^2 \boldsymbol{\alpha}\}$. Let $\mathbf{f}_2$ be the dimensionless force on the body due to $(\mathbf{u}_2, p_2)$. Then

$$\mathbf{f} = \mathbf{f}_0 + R\mathbf{f}_1 + (R^2 \log R)\,\mathbf{f}_2 + O(R^2). \tag{10.15}$$

From the definition of the Stokes resistance tensor (see paragraph preceding (7.19)), the dimensionless Stokes force on a body past which fluid streams with dimensionless velocity $\mathbf{d}$ is $d_j \phi_{ij}$. Hence

$$(f_2)_i = \tfrac{3}{320}\{31(f_0)_k \alpha_k (f_0)_j - 7(f_0)_k (f_0)_k \alpha_j\}\,\phi_{ij}. \tag{10.16}$$

Since $(f_0)_k = \alpha_l \phi_{kl}$ the latter may be written as

$$(f_2)_i = \tfrac{3}{320}\{31\phi_{ij}(\alpha_k \phi_{kl} \alpha_l) - 7\delta_{ij}(\alpha_l \phi_{kl} \phi_{km} \alpha_m)\}\,\phi_{jn}\alpha_n. \tag{10.17}$$

As may be seen by replacing $\boldsymbol{\alpha}$ by $-\boldsymbol{\alpha}$, the contribution of this term to the vector force on the body has the property that reversal of the flow at infinity merely reverses the algebraic sign of its contribution to this force.

Upon substituting (10.17) into (10.15), converting to dimensional form, and combining the result with equation (7.37), we obtain the following expression for the force on an arbitrary body to $O(R^2 \log R)$:

$$\left.\begin{aligned}
&F_i = ({}^1F)_i + ({}^2F)_i + \{\mu c U\} O(R^2),\\
&\text{where}\\
&({}^1F)_i = 6\pi\mu c U[\delta_{ij} + \tfrac{3}{16}R\{3\phi_{ij} - \delta_{ij}(U_k \phi_{kl} U_l/U^2)\}\\
&\qquad + \tfrac{3}{320}R^2 \log R\{31\phi_{ij}(U_k\phi_{kl}U_l/U^2) - 7\delta_{ij}(U_l\phi_{kl}\phi_{km}U_m/U^2)\}]\,(\phi_{jn}U_n/U),\\
&\text{and as before}\\
&({}^2F)_i = \mu c U R[(U_m U_j/U^2)\{\delta_{il} - (U_i U_l/U^2)\}]\,A_{lmj},\\
&\text{and}\\
&A_{lmj} = \frac{1}{2}\left\{\int_{S_L} ({}_{-2}v_l)_m\, dS_j + \int_{S_L} ({}_{-2}v_l)_j\, dS_m\right\} - \int_{\Gamma} (v_m)_k\,(e_l)_{kp}\,(v_j)_p\, dV.
\end{aligned}\right\} \quad (10.18)$$

Sets of conditions for which the lift force $({}^2F)_i$ is zero have been given in §8. In particular for flow along axis 1 (i.e. $U_j = \delta_{1j}U$) the drag on a body of arbitrary shape is

$$F_1 = 6\pi\mu c U[\phi_{11} + \tfrac{3}{16}R\{3\phi_{1j}\phi_{1j} - \phi_{11}^2\} + \tfrac{9}{40}(R^2\log R)(\phi_{1j}\phi_{1j}\phi_{11}) + O(R^2)].$$

Alternatively, if $\mathbf{F}_S$ is the Stokes vector force on the body due to flow along the axis 1 (i.e. $(F_S)_j = 6\pi\mu c U\phi_{1j}$), then

$$F_1 = (F_S)_1 + \tfrac{3}{16}R\frac{\{3(F_S)^2 - (F_S)_1^2\}}{6\pi\mu c U} + \tfrac{9}{40}(R^2\log R)\frac{(F_S)_1(F_S)^2}{(6\pi\mu c U)^2} + \{\mu c U\}O(R^2). \quad (10.19)$$

If, in addition, the orientation of the body is such that the stream velocity is parallel to a Stokes principal axis of resistance, the body experiences no lift force in Stokes flow, and hence $(F_S)_1 = F_S$. When this is substituted into (10.19), we obtain for the drag on the body the expression

$$\frac{F_1}{F_S} = 1 + \frac{3}{8}\left(\frac{F_S}{6\pi\mu c U}\right)R + \frac{9}{40}\left(\frac{F_S}{6\pi\mu c U}\right)^2 R^2\log R + O(R^2). \quad (10.20)$$

This agrees with Proudman & Pearson's (1957) result for the sphere ($F_S = 6\pi\mu c U$) and Breach's (1961) result for axisymmetric flow past an oblate or prolate spheroid.

An ambiguity arises in applying (10.20) or its three-dimensional analogue (10.18) which is not clearly brought out in the work of the above authors. If we replace R in (10.20) by $cU\rho/\mu$, the resulting equation becomes

$$\frac{F_1}{F_S} = 1 + \frac{F_S\rho}{16\pi\mu^2} + \frac{8}{5}\left(\frac{F_S\rho}{16\pi\mu^2}\right)^2 \log\left(\frac{cU\rho}{\mu}\right) + O(R^2).$$

Since the characteristic particle dimension c is arbitrary, the contribution of the logarithmic term to the force is not uniquely defined. This is not surprising since the R^2 term can always be written in the form $\frac{8}{5}(F_S/16\pi\mu cU)^2R^2\log k$, where k is a constant of $O(1)$ whose value must depend explicitly on the shape of the body and on the value assigned to c. Consequently (10.20) can always be expressed in the form

$$\frac{F_1}{F_S} = 1+\left(\frac{F_S}{16\pi\mu cU}\right)R+\frac{8}{5}\left(\frac{F_S}{16\pi\mu cU}\right)^2R^2\log(kR)+o(R^2). \tag{10.21}$$

From equation (10.18), it is noted that if a body is such that the force on it to $O(R)$ is reversed by a reversal of flow at infinity, then the force is in fact reversed to order $R^2\log R$. Hence if a body possesses a type of symmetry necessary to make a particular component of $^2\mathbf{F}$ zero (see §8), were we then to reverse the flow at infinity the *same component* of the vector force on the body would be reversed to order $R^2\log R$. However, the drag, given by equation (10.19), is reversed to this order, whatever the shape of the body. There is no reason to expect such general force-reversal theorems for the force to order R^2, although as pointed out in §8, a body which transforms into itself under three successive reflexions in mutually perpendicular planes necessarily possesses this force reversal property to all orders in R.

It should be noted that even for a body for which we have force reversal of some or all components of the force to $O(R^2\log R)$, we do *not* have a reversal of the velocity field in the inner expansion to this order. This may be easily seen from equation (6.12) for $\mathbf{u}_1$.

11. Summary

In this summary, the results are given in both vector-polyadic ‡(for those more familiar with this notation) and tensor notation.

Consider a fixed rigid particle of arbitrary shape, with fluid streaming past it with velocity $\mathbf{U}$. Let c be *any* characteristic particle dimension, μ be the fluid viscosity, ρ the density, $U = |\mathbf{U}|$ the magnitude of the velocity and $R = cU\rho/\mu$ the Reynolds number.

Definitions. We define the following.

(i) $\boldsymbol{\phi}(=\boldsymbol{\phi}\dagger)$ is the dimensionless Stokes resistance dyadic for the particle [$\phi_{ij}(=\phi_{ji})$ being the dimensionless Stokes resistance tensor].

(ii) $\boldsymbol{\alpha} = \mathbf{U}/U$ is a unit vector in the direction of streaming.

(iii) S_L denotes the surface of any sphere containing the particle in its interior.

(iv) Γ denotes the entire fluid space exterior to the particle.

‡ With respect to multiple products of polyads we employ the 'nesting convention' of Chapman & Cowling (1953) in conjunction with the 'sterile' or 'impotent' operation sign, $\square$, signifying no operation. Thus

$$(\mathbf{abcd})\overset{\cdot}{\underset{\square}{\cdot}}(\mathbf{efg}) = \mathbf{a}(\mathbf{b}.\mathbf{g})\,(\mathbf{c}.\mathbf{f})\,\mathbf{de} = \mathbf{ade}(\mathbf{b}.\mathbf{g})\,(\mathbf{c}.\mathbf{f}).$$

Also, the affix † denotes a transposition operator defined as follows:

$$\dagger(\mathbf{abcd}\ldots\mathbf{efgh}) = \mathbf{bacd}\ldots\mathbf{efgh}$$

and

$$(\mathbf{abcd}\ldots\mathbf{efgh})\dagger = \mathbf{abcd}\ldots\mathbf{efhg}.$$

This notation is due to Dr L. E. Scriven. Products and transpositions of polyadics may then be similarly defined, e.g.

$$\{\sum_i \mathbf{a}_i\mathbf{b}_i\mathbf{c}_i\mathbf{d}_i\}\overset{\cdot}{\underset{\square}{\cdot}}\{\sum_j \mathbf{e}_j\mathbf{f}_j\mathbf{g}_j\} = \{\sum_i\sum_j \mathbf{a}_i\mathbf{d}_i\mathbf{e}_j(\mathbf{b}_i.\mathbf{g}_j)\,(\mathbf{c}_i.\mathbf{f}_j)\}.$$

(v) $d\mathbf{S}$ is a directed element of surface area (parallel to the outer normal to the surface considered) and dV is an element of volume—these together with the operator ∇ being made dimensionless with the particle size c.

(vi) The dyadic function $\mathbf{K} = \mathbf{K}(\mathbf{r})$ [or tensor function $K_{ij}(\mathbf{r})$] ($\mathbf{r}$ = dimensionless position vector of a point relative to an origin at the centre of S_L) is a dimensionless dyadic [or tensor] field, dependent solely on the geometry of the body and the location of $\mathbf{r}$, which is defined in terms of the Stokes velocity field as follows: If fluid streams past the body with (dimensional) velocity $\mathbf{V}$ and if $\mathbf{v}'_0$ is the (dimensional) Stokes velocity field arising from this streaming flow, then

$$\mathbf{v}'_0 = \mathbf{K}.\mathbf{V},$$

or in *tensor notation*

$$(v'_0)_i = K_{ij}V_j.$$

This dyadic [or tensor] field can be determined from a knowledge of the Stokes velocity fields arising from streaming flows past the body for any three non-coplanar directions of flow. The term ${}_{-2}\mathbf{K}$ [or ${}_{-2}K_{ij}$] refers to that part of $\mathbf{K}$ [or K_{ij}] which is homogeneous in r^{-2} in its asymptotic expansion for large r. (Here, $r = |\mathbf{r}|$.)

(vii) $\mathbf{F}$ is the vector force experienced by the particle, when the fluid streams past it with velocity $\mathbf{U}$, and ${}^1\mathbf{F}$ and ${}^2\mathbf{F}$ are, respectively, those parts of $\mathbf{F}$ (to whatever order in R is being considered) which are reversed and unaltered by a reversal of the uniform flow at infinity.

Results. The main results may be expressed as follows:

1. To $O(R)$, $\mathbf{F}$ is given by the relation

$$\mathbf{F} = {}^1\mathbf{F} + {}^2\mathbf{F} + \{\mu c U\} o(R), \tag{11.1}$$

where *in vector-polyadic notation* ${}^1\mathbf{F}$ and ${}^2\mathbf{F}$ are given by the expressions

$${}^1\mathbf{F} = 6\pi\mu c U[\boldsymbol{\phi}.\boldsymbol{\alpha} + \tfrac{3}{16}R\{3(\boldsymbol{\phi}.\boldsymbol{\phi}.\boldsymbol{\alpha}) - (\boldsymbol{\alpha}.\boldsymbol{\phi}.\boldsymbol{\alpha})(\boldsymbol{\phi}.\boldsymbol{\alpha})\}] \tag{11.2a}$$

and

$${}^2\mathbf{F} = \mu c U R(\mathbf{I} - \boldsymbol{\alpha}\boldsymbol{\alpha}).\mathbf{A} : \boldsymbol{\alpha}\boldsymbol{\alpha}, \tag{11.3a}$$

in which $\mathbf{I}$ is the dyadic idemfactor, and $\mathbf{A}$ ($= \mathbf{A}\dagger$) is a dimensionless triadic, dependent solely on the geometry of the particle, which is defined in terms of the Stokes velocity field by:

$$\mathbf{A} = \frac{1}{2}\int_{S_L} [({}_{-2}\mathbf{K}^\dagger)\,d\mathbf{S} + \{({}_{-2}\mathbf{K}^\dagger)\,d\mathbf{S}\}\dagger] - \frac{1}{2}\int_{\Gamma} [\{(\nabla\mathbf{K}) + {}^\dagger(\nabla\mathbf{K})\} \dot{\underset{\square}{}} \{(\mathbf{K}^\dagger)\,\mathbf{K}\}]\,dV; \tag{11.4a}$$

in tensor notation ${}^1\mathbf{F}$ and ${}^2\mathbf{F}$ are given by

$$({}^1F)_i = 6\pi\mu c U[\delta_{ij} + \tfrac{3}{16}R\{3\phi_{ij} - \delta_{ij}(\alpha_k\phi_{kl}\alpha_l)\}]\,\phi_{jm}\alpha_m \tag{11.2b}$$

and

$$({}^2F)_i = \mu c U R[\alpha_m\alpha_j(\delta_{il} - \alpha_i\alpha_l)]\,A_{lmj}, \tag{11.3b}$$

in which A_{lmj} ($= A_{ljm}$) is a dimensionless third-order tensor, dependent solely on the geometry of the particle, which is defined in terms of the Stokes velocity field by

$$A_{lmj} = \frac{1}{2}\int_{S_L} \{({}_{-2}K_{ml})\,dS_j + ({}_{-2}K_{jl})\,dS_m\} - \frac{1}{2}\int_{\Gamma} K_{km}K_{pj}(K_{kl,p} + K_{pl,k})\,dV; \tag{11.4b}$$

an alternative form of this equation is

$$A_{lmj} = \frac{1}{2}\left\{\int_{S_L} ({}_{-2}v_l)_m\, dS_j + ({}_{-2}v_l)_j\, dS_m\right\} - \int_{\Gamma} (v_m)_k\,(e_l)_{kp}\,(v_j)_p\, dV, \qquad (11.4c)$$

in which $(v_i)_j$ is the jth component of the Stokes velocity field due to a uniform stream at infinity of velocity unity in the ith direction, $(e_i)_{jk}$ is the (j, k) component of the rate-of-strain tensor of the same Stokes velocity field, and $({}_{-2}v_i)_j$ is the term homogeneous in r^{-2} in the asymptotic expansion of $(v_i)_j$ for large r.

2. The 'irreversible' force ${}^2\mathbf{F}$ is a *lift* force. When the body possesses certain symmetry properties (summarized in §8), then some or all components of ${}^2\mathbf{F}$ are zero.

3. ${}^2\mathbf{F}$ is *not* always zero.

4. The vector force $\overline{\mathbf{F}}$ on a particle of arbitrary shape, as calculated by the classical Oseen equations, is, to $O(R)$,

$$\overline{\mathbf{F}} = {}^1\overline{\mathbf{F}} + {}^2\overline{\mathbf{F}} + \{\mu c U\}\, o(R), \qquad (11.5)$$

where

$${}^1\overline{\mathbf{F}} = {}^1\mathbf{F}; \qquad (11.6)$$

in vector-polyadic notation ${}^2\overline{\mathbf{F}}$ is given by

$${}^2\overline{\mathbf{F}} = \mu c U R(\mathbf{I} - \boldsymbol{\alpha}\boldsymbol{\alpha}).\bar{\mathbf{A}} : \boldsymbol{\alpha}\boldsymbol{\alpha}, \qquad (11.7a)$$

where $\bar{\mathbf{A}}$ is a triadic (different from $\mathbf{A}$) given in terms of the Stokes velocity field by the expression

$$\mathbf{A} = \frac{1}{2}\int_{S_L} [({}_{-2}\mathbf{K}^\dagger)\, dS + \{({}_{-2}\mathbf{K}^\dagger)\, d\mathbf{S}\}^\dagger] - \frac{1}{2}\int_{\Gamma} \langle {}^\dagger\{(\nabla\mathbf{K})^\dagger . \mathbf{K}\} + [{}^\dagger\{(\nabla\mathbf{K})^\dagger . \mathbf{K}\}]^\dagger\rangle\, dV; \qquad (11.8a)$$

in tensor notation ${}^2\overline{\mathbf{F}}$ is given by

$$({}^2\bar{F})_i = \mu c U R[\alpha_m \alpha_j(\delta_{il} - \alpha_i\alpha_l)]\, \bar{A}_{lmj}, \qquad (11.7b)$$

where $\bar{A}_{lmj}$ is a third-order tensor (different from A_{lmj}) given in terms of the Stokes velocity field by

$$\bar{A}_{lmj} = \frac{1}{2}\int_{S_L} \{({}_{-2}K_{ml})\, dS_j + ({}_{-2}K_{jl})\, dS_m\} - \frac{1}{2}\int_{\Gamma} \{K_{km} K_{kl,j} + K_{kj} + K_{lj} K_{kl,m}\}\, dV, \qquad (11.8b)$$

or alternatively by

$$\bar{A}_{lmj} = \frac{1}{2}\int_{S_L} \{({}_{-2}v_l)_m\, dS_j + ({}_{-2}v_l)_j\, dS_m\} - \frac{1}{2}\int_{\Gamma} \{(v_m)_k\,(v_l)_{k,j} + (v_j)_k\,(v_l)_{k,m}\}\, dV. \qquad (11.8c)$$

5. The 'irreversible' force ${}^2\overline{\mathbf{F}}$ is a *lift* force. When the body possesses certain symmetry properties (summarized in §8) sufficient to make a particular component of ${}^2\mathbf{F}$ zero, then the same component of ${}^2\overline{\mathbf{F}}$ is zero.

6. ${}^2\overline{\mathbf{F}}$ is *not* always zero, and *not* always the same as ${}^2\mathbf{F}$.

7. The solution for $\mathbf{F}$ to order $R^2 \log R$ is

$$\mathbf{F} = {}^1\mathbf{F} + {}^2\mathbf{F} + \{\mu c U\}\, O(R^2), \qquad (11.9)$$

where ${}^2\mathbf{F}$ is the same as that given in equations (11.3*a*, *b*). ${}^1\mathbf{F}$ has an additional term of order $R^2 \log R$ added, and is now given *in vector-polyadic notation* by the expression

$$ {}^1\mathbf{F} = 6\pi\mu c U[\boldsymbol{\phi}.\boldsymbol{\alpha} + \tfrac{3}{16}R\{3(\boldsymbol{\phi}.\boldsymbol{\phi}.\boldsymbol{\alpha}) - (\boldsymbol{\alpha}.\boldsymbol{\phi}.\boldsymbol{\alpha})(\boldsymbol{\phi}.\boldsymbol{\alpha})\} + \tfrac{3}{320}(R^2\log R)\{31(\boldsymbol{\phi}.\boldsymbol{\phi}.\boldsymbol{\alpha})(\boldsymbol{\alpha}.\boldsymbol{\phi}.\boldsymbol{\alpha}) - 7(\boldsymbol{\alpha}.\boldsymbol{\phi}.\boldsymbol{\phi}.\boldsymbol{\alpha})(\boldsymbol{\phi}.\boldsymbol{\alpha})\}] \quad (11.10a) $$

or *in tensor notation* by

$$ ({}^1F)_i = 6\pi\mu c U[\delta_{ij} + \tfrac{3}{16}\{3\phi_{ij} - \delta_{ij}(\alpha_k\phi_{kl}\alpha_l)\} + \tfrac{3}{320}(R^2\log R)\{31\phi_{ij}(\alpha_k\phi_{kl}\alpha_l) - 7\delta_{ij}(\alpha_l\phi_{kl}\phi_{kn}\alpha_n)\}]\phi_{jm}\alpha_m. \quad (11.10b) $$

However, the additional term of $O(R^2 \log R)$ in this solution is not unique, since it is not possible to separate the terms involving (R^2) and $(R^2 \log R)$ in a unique manner. Perhaps the major conclusion of the present investigation is that the Oseen linearization scheme does not, in general, yield the correct vector force on a body to $O(R)$.

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Appendix 1

We consider here various types of symmetry for bodies which imply that a particular component of the tensor A_{lmj} is zero.

Suppose $l = m = j$; then

$$ \begin{aligned} A_{lll} &= \int_{S_L} ({}_{-2}v_l)_l\, dS_l - \sum_{k,p} \int_\Gamma (v_l)_k (e_l)_{kp} (v_l)_p\, dV \quad \text{(no summation over } l) \\ &= \int_{S_L} ({}_{-2}v_l)_l\, dS_l - \sum_p \int_\Gamma \{\tfrac{1}{2}(v_l)^2 (v_l)_p\}_{,p}\, dV \\ &= \int_{S_L} ({}_{-2}v_l)_l\, dS_l - \sum_p \lim_{L\to\infty} \int_{S_L} \tfrac{1}{2}(v_l)^2 (v_l)_p\, dS_p. \end{aligned} $$

Using the same argument as was used in deriving equation (7.27), we obtain

$$ A_{lll} = \int_{S_L} ({}_{-2}v_l)_l\, dS_l - \int_{S_L} ({}_{-2}v_l)_l\, dS_l = 0. $$

Thus if $l = m = j$, $A_{lmj} = 0$.

Let the axes $(0_1, 0_2, 0_3)$ be axes fixed in the body and in space. Now consider another set of axes $(0_{1'}, 0_{2'}, 0_{3'})$, coincident initially with $(0_1, 0_2, 0_3)$, which we rotate and/or reflect in such a manner that each of the axes $0_{1'}, 0_{2'}, 0_{3'}$, lie along one of the axes $0_1, 0_2, 0_3$ or their continuations, $0_{-1}, 0_{-2}, 0_{-3}$, respectively, in their negative directions. We may represent such a rotation by $(\alpha_1, \alpha_2, \alpha_3)$ where $\alpha_1, \alpha_2, \alpha_3$ are the labels of the fixed axes into which $0_{1'}, 0_{2'}, 0_{3'}$ transform, e.g. $(1, -2, -3)$ is a rotation through an angle π about the 0_1 axis.

Consider the class of bodies whose surface B when specified in terms of the variables x_1, x_2, x_3 (Cartesian co-ordinates defined by the axes $0_1, 0_2, 0_3$) is of the same form as when it is specified in terms of x_1', x_2', x_3' (defined by the axes $0_{1'}, 0_{2'}, 0_{3'}$ *after* the rotation), i.e. the surface B is $f(x_1, x_2, x_3) = 0$ or $f(x_1', x_2', x_3') = 0$. Now suppose the transformation from (x_1, x_2, x_3) variables to (x_1', x_2', x_3') variables is given by

$$x_i' = \alpha_{ij} x_j. \tag{A 1.1}$$

From equation (7.37),

$$A_{lmj} = \int_{S_L} C_{lmj} - \int_\Gamma D_{lmj}\, dV, \tag{A 1.2}$$

where C_{lmj} and D_{lmj} are third-order tensors given by

$$C_{lmj} = \tfrac{1}{2}\{({}_{-2}v_l)_m\, dS_j + ({}_{-2}v_l)_j\, dS_m\}, \tag{A 1.3}$$

$$D_{lmj} = (v_m)_k\, (e_l)_{kp}\, (v_j)_p. \tag{A 1.4}$$

Suppose that C_{lmj} and D_{lmj} are transformed respectively to C'_{lmj} and D'_{lmj}. Now, since the body is similarly situated with respect to both the old and new axes, and since Stokes equations are invariant under rotation and reflexion, we must have

$$C'_{pqr}(\mathbf{x}') = C_{pqr}(\mathbf{x}). \tag{A 1.5}$$

Also by the tensor transformation formula

$$C'_{pqr}(\mathbf{x}') = \alpha_{pl}\alpha_{qm}\alpha_{rj} C_{lmj}(\mathbf{x}'). \tag{A 1.6}$$

Hence

$$C_{pqr}(\mathbf{x}) = \alpha_{pl}\alpha_{qm}\alpha_{rj} C_{lmj}(\mathbf{x}'). \tag{A 1.7}$$

From the definition of the transformation $(\alpha_1, \alpha_2, \alpha_3)$, the transformation (A 1.1) is

$$x_i' = \sum_j \{\delta_{i|\alpha_j|}\operatorname{sgn}(\alpha_j)\} x_j.$$

Therefore

$$\alpha_{ij} = \{\operatorname{sgn}(\alpha_j)\}\,\delta_{i|\alpha_j|}. \tag{A 1.8}$$

Substituting in equation (A 1.7), we obtain

$$C_{pqr}(\mathbf{x}) = \{\operatorname{sgn}(\alpha_l\alpha_m\alpha_j)\}\,\delta_{p|\alpha_l|}\delta_{q|\alpha_m|}\delta_{r|\alpha_j|} C_{lmj}(\mathbf{x}').$$

Thus

$$C_{|\alpha_l||\alpha_m||\alpha_j|}(\mathbf{x}) = \{\operatorname{sgn}(\alpha_l\alpha_m\alpha_j)\} C_{lmj}(\mathbf{x}'). \tag{A 1.9a}$$

From equation (A 1.3), C_{lmj} is symmetric in m, j.

Therefore

$$C_{|\alpha_l||\alpha_m||\alpha_j|}(\mathbf{x}) = \{\operatorname{sgn}(\alpha_l\alpha_m\alpha_j)\} C_{ljm}(\mathbf{x}'). \tag{A 1.9b}$$

Similarly for D_{lmj} we obtain

$$D_{|\alpha_l||\alpha_m||\alpha_j|}(\mathbf{x}) = \{\operatorname{sgn}(\alpha_l\alpha_m\alpha_j)\} D_{lmj}(\mathbf{x}'), \tag{A 1.10a}$$

$$D_{|\alpha_l||\alpha_m||\alpha_j|}(\mathbf{x}) = \{\operatorname{sgn}(\alpha_l\alpha_m\alpha_j)\} D_{ljm}(\mathbf{x}'). \tag{A 1.10b}$$

That D_{lmj} is symmetric in m, j can be easily proved as follows:

$$D_{lmj} = (v_m)_k\,(e_l)_{kp}\,(v_j)_p = (v_m)_p\,(e_l)_{kp}\,(v_j)_k = (v_j)_k\,(e_l)_{kp}\,(v_m)_p = D_{ljm}.$$

From equations (A 1.9a), (A 1.9b) it is seen that

$$\int_{S_L} (C_{lmj}) = 0$$

if *either* $\quad i = |\alpha_i| \quad$ and $\quad \operatorname{sgn}(\alpha_l\alpha_m\alpha_j) = -1,$

or $\quad l = |\alpha_l|,\ m = |\alpha_j|,\ j = |\alpha_m| \quad$ and $\quad \operatorname{sgn}(\alpha_l\alpha_m\alpha_j) = -1.$

It is seen that if either of these conditions holds then $\int_\Gamma (D_{lmj})\,dV = 0$, the inner boundary of Γ (i.e. the surface B of the body) being of the same form in either the dashed or undashed system of co-ordinates.

Hence from equation (A 1.2), A_{lmj} is zero if

either $$i = |\alpha_i| \quad \text{and} \quad \operatorname{sgn}(\alpha_l \alpha_m \alpha_j) = -1 \tag{A 1.11a}$$

or $$l = |\alpha_l|,\ m = |\alpha_j|,\ j = |\alpha_m| \quad \text{and} \quad \operatorname{sgn}(\alpha_l \alpha_m \alpha_j) = -1. \tag{A 1.11b}$$

We have already shown that $A_{lmj} = 0$ if $l = m = j$. We shall now deal with various other cases which may occur.

Case 1, $l = m \neq j$, or $l = j \neq m$ (because of symmetry). For this we cannot have condition (A 1.11*b*) holding, since this would imply $|\alpha_1| = |\alpha_j|$ or $l = j$. Condition (A 1.11*a*) shows that $A_{llj} = 0$ if $i = |\alpha_i|$ and $\operatorname{sgn}(\alpha_j) = -1$, e.g. $A_{112} = 0$ if $(\alpha_1, \alpha_2, \alpha_3)$ has values:

$$(1, -2, 3); \quad (1, -2, -3); \quad (-1, -2, 3); \quad (-1, -2, -3).$$

Case 2, $l \neq m = j$. For this, conditions (A 1.11*a*) and (A 1.11*b*) become identical. Therefore
$$A_{ljj} = 0 \quad \text{if} \quad i = |\alpha_i| \quad \text{and} \quad \operatorname{sgn}(\alpha_l) = -1.$$

Case 3, *l, m, j all different.* Then condition (A 1.11*a*) gives $A_{lmj} = 0$ for the following values of $(\alpha_1\, \alpha_2\, \alpha_3)$:

$$(-1, -2, -3); \quad (-1, 2, 3); \quad (1, -2, 3); \quad (1, 2, -3).$$

Also the condition (A 1.11*b*) gives A_{123} or A_{132} zero for the following values of $(\alpha_1 \alpha_2 \alpha_3)$:
$$(1, 3, -2); \quad (1, -3, 2); \quad (-1, 3, 2); \quad (-1, -3, -2),$$

with similar sets of transformations for A_{231} and A_{312}; $(-1, 3, 2)$ is a combination of transformations $(1, 3, -2)$, $(1, -2, 3)$ and $(-1, -2, -3)$ applied in succession, these being transformations already included for $A_{123} = 0$. A similar result holds for $(-1, -3, -2)$. Thus we may omit the transformations $(-1, 3, 2)$ and $(-1, -3, -2)$ from our list.

All the transformations mentioned in the above three cases are of the following four types.

(*a*) A rotation about an axis through an angle π.

(*b*) A rotation about an axis through an angle $\frac{1}{2}\pi$.

(*c*) A reflexion about a plane formed by two axes.

(The plane through axes 2 and 3 will be labelled 1.)

(*d*) A simultaneous reflexion in all three such planes.

For a particular $[lmj]$ we have found those transformations of the above four types that, when applied to the co-ordinates leave the form of the particle surface B unchanged, would imply that A_{lmj} is zero. We may equivalently consider the transformation as transforming the body into itself, rather than a transformation of axes. We present in table 1, for each value of $[lmj]$, a list of those transformations of the types (*a*) to (*d*) given above, for which, if the body is such that any one of these transformations transforms the body into itself, one may conclude that A_{lmj} is zero.

Now from equation (7.37)

$$({}^2F)_i = \mu c U R[(U_m U_j/U^2)\{\delta_{il} - (U_i U_l/U^2)\}] A_{lmj}.$$

By the use of table 1, together with this formula, we shall now find what types of symmetry a body will require for some or all the components of $({}^2F)_i$ to vanish for either a general or a particular value of U_i. There are several separate cases which we shall consider.

Values of [*lmj*]	Rotation through an angle π	Rotation through an angle $\frac{1}{2}\pi$	Reflexion	Triple reflexion
	Axes	Axes	Planes	
[112] [121] [211]	1, 3	—	2	Yes
[113] [131] [311]	1, 2	—	3	Yes
[221] [212] [122]	2, 3	—	1	Yes
[223] [232] [322]	1, 2	—	3	Yes
[331] [313] [133]	2, 3	—	1	Yes
[332] [323] [233]	1, 3	—	2	Yes
[123] [132]	—	1	1, 2, 3	Yes
[231] [213]	—	2	1, 2, 3	Yes
[312] [321]	—	3	1, 2, 3	Yes
[111] [222] [333]	1, 2, 3	1, 2, 3	1, 2, 3	Yes

TABLE 1

(I) $({}^2F)_i$ *to vanish whatever the direction of* **U**

For this we require $A_{lmj} = 0$ for all l, m, j. From table 1, this will happen if the body goes into itself under either of the following sets of transformations when account is taken of redundant cases:

(i) Triple reflexion,
(ii) Rotations through $\frac{1}{2}\pi$ about axes 1, 2 and 3. (A 1.12)

(II) $({}^2F)_i$ *to vanish for* **U** *lying in plane* 3

If **U** is $(U_1, U_2, 0)$, then for $({}^2F)_i$ to vanish we only require A_{lmj} to be zero for the following sets of values for [*lmj*]: [112], [122], [211], [212], [311], [312], [322] together with those formed by interchanging m and j.

This follows if the body goes into itself under the following set of transformations, together with those given in case I

Rotation through π about axis 1. Rotation $\frac{1}{2}\pi$ about axis 3. (A 1.13)

(III) $({}^2F)_3$ *to vanish if* **U** *lies in plane* 3

This happens if $A_{lmj} = 0$ for the following sets of [*lmj*]: [311], [312], [322] and those with m and j interchanged. It can be easily shown that this follows if the body goes into itself under the following transformation (not included in previous cases)

Reflexion in plane (3). (A 1.14)

(IV) $({}^2F)_1$ and $({}^2F)_2$ to vanish if **U** *lies in plane* 3

This happens if $A_{lmj} = 0$ for the following sets of $[lmj]$: [112], [122], [211], [212] and those with m and j interchanged.

The relevant sets of transformations (not included in previous cases) reduce to

Rotation through π about axis 3. (A 1.15)

(V) $({}^2F)_i$ to vanish if **U** *lies along the axis* 1

$({}^2F)_1$ must vanish for all bodies for $\mathbf{U} = (U_1, 0, 0)$, since this is the contribution of $({}^2\mathbf{F})$ to the scalar drag. However for $({}^2\mathbf{F})$ to vanish requires A_{lmj} to be zero for $[lmj]$ equal to [211] or [311].

The relevant sets of transformations (not included in previous cases) reduce to

Rotation through π about axis 1. (A 1.16)

(VI) $({}^2F)_1$ and $({}^2F)_2$ to vanish if **U** *lies along the axis* 1

$({}^2F)_1$ vanishes for all bodies, and $({}^2F)_2$ vanishes if $A_{lmj} = 0$ for $[lmj]$ equal to [211].

The relevant sets of transformations (not included in previous cases) reduce to

Reflexion in plane 2. (A 1.17)

It is noted that this case is really a combination of the result (A 1.14) of case (III) together with the result that the scalar drag component of $({}^2F)_i$ on a body must vanish.

Appendix 2

In this appendix, we shall show by considering a particular example that ${}^2\mathbf{F}$ and ${}^2\mathbf{F} - {}^2\overline{\mathbf{F}}$ are not always zero. This will involve demonstrating that $A_{lmj} \neq 0$ and $A_{lmj} - \bar{A}_{lmj} \neq 0$ for some value of l, m, j.

Consider the uniform streaming past a slightly deformed sphere whose surface is defined in spherical polar co-ordinates by

$$r = 1 + \epsilon\beta_k(\theta, \phi), \tag{A 2.1}$$

where all lengths have been made dimensionless by the radius of the undeformed sphere. $\beta_k(\theta, \phi)$ is a surface harmonic of order k which, for our particular example, we will take to be $\beta_3(\theta, \phi) = r^4(\partial^3/\partial r_3^3)(1/r)$, which is seen to be equal to $\partial^3/\partial r_3^3\,(1/r)$ evaluated for $r = 1$. ϵ is to be taken so small that squares and higher-order terms may be neglected.

Let $\mathbf{v}$ be the Stokes flow around such a body due to a uniform velocity $\mathbf{W}$ at infinity. Then

$$\nabla^2\mathbf{v} = \mu^{-1}\nabla p, \quad \nabla . \mathbf{v} = 0, \tag{A 2.2}$$

with the boundary conditions

$$\left.\begin{aligned} &\mathbf{v} = 0 \quad \text{on} \quad r = 1 + \epsilon\left[\frac{\partial^3}{\partial r_3^3}\left(\frac{1}{r}\right)\right]_{r=1}, \\ &\mathbf{v} \to \mathbf{W} \quad \text{as} \quad r \to \infty. \end{aligned}\right\} \tag{A 2.3}$$

These equations have a solution of the form

$$\left.\begin{aligned} \mathbf{v} &= \mathbf{v}_s + \epsilon \mathbf{v}_p + O(\epsilon^2), \\ p &= p_s + \epsilon p_p + O(\epsilon^2), \end{aligned}\right\} \tag{A 2.4}$$

where $(\mathbf{v}_s, p_s)$ is the solution of the Stokes equations for streaming flow past a *sphere* of radius unity and $(\mathbf{v}_p, p_p)$ is a perturbation field which satisfies Stokes equations. $(\mathbf{v}_s, p_s)$ may be easily shown to be given by

$$\left.\begin{aligned} (v_s)_i &= W_i(1 - \tfrac{3}{4}r^{-1} - \tfrac{1}{4}r^{-3}) - W_k r_k r_i \tfrac{3}{4}(r^{-3} - r^{-5}), \\ \mu^{-1} p_s &= -\tfrac{3}{2} r^{-3} W_k r_k; \end{aligned}\right\} \tag{A 2.5}$$

$(\mathbf{v}_p, p_p)$ is given by Lamb's (1932, p. 594) general solution of Stokes equations as

$$\left.\begin{aligned} \mathbf{v}_p &= \sum_{n=0}^{\infty} \left[\nabla \wedge (\mathbf{r}\chi_{-(n+1)}) + \nabla \phi_{-(n+1)} - \frac{(n-2)}{2n(2n-1)} r^2 \nabla \left(\frac{p_{-(n+1)}}{\mu}\right) + \mathbf{r} \frac{(n+1)}{n(2n-1)} \left(\frac{p_{-(n+1)}}{\mu}\right) \right], \\ p_p &= \sum_{n=0}^{\infty} p_{-(n+1)}, \end{aligned}\right\} \tag{A 2.6}$$

where

$$\left.\begin{aligned} &\frac{1}{\mu} p_{-3} = \tfrac{3}{7} r^{-5} (\mathbf{W}.\nabla) \left\{ r^7 \frac{\partial^3}{\partial r_3^3} \left(\frac{1}{r}\right) \right\}, \quad \frac{1}{\mu} p_{-5} = \tfrac{3}{2} (\mathbf{W}.\nabla) \left\{ \frac{\partial^3}{\partial r_3^3} \left(\frac{1}{r}\right) \right\}, \\ &\frac{1}{\mu} p_{-n} = 0 \quad \text{if} \quad n \neq 3, 5; \end{aligned}\right\} \tag{A 2.7}$$

$$\left.\begin{aligned} &\phi_{-3} = \tfrac{1}{14} r^{-5} (\mathbf{W}.\nabla) \left\{ r^7 \frac{\partial^3}{\partial r_3^3} \left(\frac{1}{r}\right) \right\}, \quad \phi_{-5} = \tfrac{3}{28} (\mathbf{W}.\nabla) \left\{ \frac{\partial^3}{\partial r_3^3} \left(\frac{1}{r}\right) \right\}, \\ &\phi_{-n} = 0 \quad \text{if} \quad n \neq 3, 5; \end{aligned}\right\} \tag{A 2.8}$$

$$\chi_{-4} = \tfrac{1}{8} r^{-4} \mathbf{W}.\nabla \wedge \left\{ \mathbf{r} r^4 \frac{\partial^3}{\partial r_3^3} \left(\frac{1}{r}\right) \right\}, \quad \chi_{-n} = 0 \quad \text{if} \quad n \neq 4. \tag{A 2.9}$$

It may be verified that this solution for $(\mathbf{v}_p, p_p)$ is correct, by showing (by direct substitution) that the boundary conditions (A 2.3) are satisfied to order ϵ.

Writing

$$H(a_1 a_2 \ldots a_n) \equiv \frac{\partial^n}{\partial r_{a_1} \partial r_{a_2} \ldots \partial r_{a_n}} \left(\frac{1}{r}\right),$$

it may be shown that if $(v_{p\lambda})_i$ is the ith component of $\mathbf{v}_p$ for $W_j = \delta_{j\lambda}$ then

$$\begin{aligned} (v_{p\lambda})_i = {} & \tfrac{1}{8} \epsilon_{ijk} \epsilon_{\lambda lm} [\delta_{mj} r_k H(l333) + r_m r_k H(jl333)] \\ & + \tfrac{1}{56}(r^2 + 6) H(i\lambda 333) + \tfrac{1}{56}(12r^2 + 23) r_i H(\lambda 333) \\ & + \tfrac{1}{2}[\delta_{i\lambda} H(333) + r_\lambda H(i333) + 3 r_i r_\lambda H(333)], \end{aligned} \tag{A 2.10}$$

where the results $\epsilon_{\lambda lm} \delta_{lm} = 0$ and $\epsilon_{\lambda lm} r_l r_m = 0$ have been used. Differentiating with respect to r_n we obtain

$$\begin{aligned} (v_{p\lambda})_{i,n} = {} & \tfrac{1}{8} \epsilon_{ijk} \epsilon_{\lambda lm} [\delta_{mj} \delta_{kn} H(l333) + \delta_{mj} r_k H(nl333) \\ & + \delta_{mn} r_k H(jl333) + \delta_{kn} r_m H(jl333) + r_m r_k H(njl333)] \\ & + \tfrac{1}{56}(r^2 + 6) H(ni\lambda 333) + \tfrac{1}{28} r_n H(i\lambda 333) + \tfrac{1}{56}(12r^2 + 23) r_i H(n\lambda 333) \\ & + \tfrac{1}{56}(12r^2 + 23) \delta_{in} H(\lambda 333) + \tfrac{3}{7} r_n r_i H(\lambda 333) + \tfrac{1}{2}[\delta_{i\lambda} H(n333) \\ & + r_\lambda H(ni333) + \delta_{\lambda n} H(i333) + 3 r_i r_\lambda H(n333) + 3\delta_{in} r_\lambda H(333) \\ & + 3\lambda_{\lambda n} r_i H(333)]. \end{aligned} \tag{A 2.11}$$

Defining $(v_{s\nu})_i$ in a manner similar to that for $(v_{p\lambda})_i$, we see that it is given by

$$(v_{s\nu})_i = \delta_{\nu i}(1-\tfrac{3}{4}r^{-1}-\tfrac{1}{4}r^{-3})-r_i r_\nu \tfrac{3}{4}(r^{-3}-r^{-5}). \qquad \text{(A 2.12)}$$

Differentiating with respect to r_n gives

$$(v_{s\nu})_{i,n} = \tfrac{3}{4}[\delta_{\nu i} r_n(r^{-3}+r^{-5})-\delta_{in} r_\nu(r^{-3}-r^{-5})-\delta_{\nu n} r_i(r^{-3}-r^{-5}) - r_i r_\nu r_n(-3r^{-5}+5r^{-7})]. \qquad \text{(A 2.13)}$$

Therefore

$$(e_{s\nu})_{in} = \tfrac{3}{4}[\delta_{\nu i} r_n r^{-5}+\delta_{\nu n} r_i r^{-5}-\delta_{in} r_\nu(r^{-3}-r^{-5})-r_i r_\nu r_n(-3r^{-5}+5r^{-7})], \quad \text{(A 2.14)}$$

$(e_{s\nu})_{in}$ being the (i, n) component of the rate-of-strain tensor corresponding to the flow field $\mathbf{v}_s$ for which $W_j = \delta_{j\nu}$.

We will now proceed to calculate the value of A_{311} to order ϵ. From equation (7.37)

$$A_{311} = \int_{S_L} ({}_{-2}v_3)_1 dS_1 - \int_\Gamma (v_1)_i (e_3)_{in} (v_1)_n dV, \qquad \text{(A 2.15)}$$

where Γ is the volume exterior to the deformed sphere. It is seen that since we wish to calculate A_{311} only to order ϵ, we may take Γ to be the volume exterior to the *sphere* $r = 1$, since the integrand $(v_1)_i (e_3)_{in} (v_1)_n$ vanishes on the surface of the deformed sphere. From §8 we know that for the *sphere* $r = 1$ the value of A_{311} must be zero, i.e.

$$O = \int_{S_L} ({}_{-2}v_{s3})_1 dS_1 - \int_\Gamma (v_{s1})_i (e_{s3})_{in} (v_{s1})_n dV. \qquad \text{(A 2.16)}$$

Substituting for $\mathbf{v}$ from equations (A 2.4) into equation (A 2.15) and using equation (A 2.16) we obtain

$$A_{311} = \epsilon\left[\int_{S_L} ({}_{-2}v_{p3})_1 dS_1 - 2\int_\Gamma (v_{p1})_i (e_{s3})_{in} (v_{s1})_n dV - \int_\Gamma (v_{s1})_i (e_{p3})_{in} (v_{s1})_n dV\right] + O(\epsilon^3). \qquad \text{(A 2.17)}$$

In order to evaluate these integrals we will need to use the following four lemmas:

Lemma 1, u and t are positive integers and w is any number such that $w < u-t$. Then, if A_t is a homogeneous polynomial in the r_i of degree t,

$$\int_\Gamma r^w A_t H(a_1 a_2 \ldots a_u)\, dV = 0 \quad \text{if} \quad t < u.$$

Proof. The condition $w < u-t$ ensures the convergence of the integral. The result is immediate by the orthogonality property of spherical harmonics since

$$H(a_1 a_2 \ldots a_u) = r^{-u-1} \quad \text{(surface harmonic of order } u)$$

and A_t can be expressed as

$$A_t = r^t \quad \text{(sum of surface harmonics of order} \leqslant t).$$

Lemma 2, $r_i H(ia_1 a_2 \dots a_u) = -H(a_1 a_2 \dots a_u)$.

Proof

$$H(a_1 a_2 \dots a_u) = r^{-u-1} \quad \text{(homogeneous polynomial of degree } u \text{ in } r_i)$$
$$\equiv r^{-u-1} B_u \text{ say.}$$

Hence
$$r_i H(ia_1, a_2 \dots a_u) = -r^{-u-1}(u+1) B_u + r^{-u-1} r_i \frac{\partial}{\partial r_i} B_u$$
$$= -r^{-u-1} B_u = -H(a_1 a_2 \dots a_u).$$

Lemma (3), $r_i r_i = r^2$.

Lemma (4), $\epsilon_{ijk} r_j r_k = 0$.

Consider the $\int_{S_L} ({}_{-2}v_{p3})_1 dS_1$ which appears in the equation (A 2.17).

$$({}_{-2}v_{p3})_1 = [\tfrac{3}{14} r^2 r_1 H(3333) + \tfrac{3}{2} r_1 r_3 H(333)] \quad \text{from equation (A 2.10).}$$

Hence
$$\int_{S_L} ({}_{-2}v_{p3})_1 dS_1 = \int_{S_L} \{\tfrac{3}{14} r r_1^2 H(3333) + \tfrac{3}{2} r^{-1} r_1^2 r_3 H(333)\} dS,$$

which by lemma (1) gives

$$\int_{S_L} ({}_{-2}v_{p3})_1 dS_1 = \tfrac{3}{2} J, \tag{A 2.18}$$

where
$$J = \int_{S_L} \frac{r_1^2 r_3}{r} \frac{\partial^3}{\partial r_3^3} \left(\frac{1}{r}\right) dS. \tag{A 2.19}$$

Consider now $\int_\Gamma (v_{p1})_i (e_{s3})_{in} (v_{s1})_n dV$. This is seen to be

$$\int_\Gamma [\tfrac{1}{8} \epsilon_{ijk} \epsilon_{1lm} \{\delta_{mj} r_k H(l333) + r_m r_k H(jl333)\} + \tfrac{1}{56}(r^2+6) H(i\lambda 333)$$
$$+ \tfrac{1}{56}(12r^2+23) r_i H(\lambda 333) + \tfrac{1}{2}\{\delta_{i1} H(333) + r_1 H(i333) + 3 r_i r_1 H(333)\}]$$
$$\times \tfrac{3}{4}[\delta_{3i} r_n r^{-5} + \delta_{3n} r_i r^{-5} - \delta_{in} r_3 (r^{-3} - r^{-5}) - r_i r_n r_3 (-3r^{-5} + 5r^{-7})]$$
$$\times [\delta_{1n}(1 - \tfrac{3}{4} r^{-1} - \tfrac{1}{4} r^{-3}) - r_1 r_n \tfrac{3}{4}(r^{-3} - r^{-5})] dV. \tag{A 2.20}$$

Using the lemmas 1 to 4 given above, we see that most of the terms in this expression are zero. Hence

$$\int_\Gamma (v_{p1})_i (e_{s3})_{in} (v_{s1})_n dV$$
$$= \frac{9}{32} \int_\Gamma H(333) . (r_1^2 r_3) dV (8r^{-3} - 12r^{-4} - 12r^{-5} + 19r^{-6} - 3r^{-8}).$$

Performing the radial integration we obtain

$$\int_\Gamma (v_{p1})_i (e_{s3})_{in} (v_{s1})_n dV = \tfrac{81}{128} J. \tag{A 2.21}$$

In a similar manner we consider

$$\int_\Gamma (v_{s1})_i (e_{p3})_{in} (v_{s1})_n dV = \int_\Gamma (v_{s1})_i (v_{p3})_{i,n} (v_{s1})_n dV,$$

which is seen to be equal to

$$\int_\Gamma [\tfrac{1}{8}\epsilon_{ijk}\epsilon_{3lm}\{\delta_{mj}\delta_{kn}H(l333)+\delta_{mj}r_kH(nl333)+\delta_{mn}r_kH(jl333) \\ +\delta_{kn}r_mH(jl333)+r_mr_kH(njl333)\}+\tfrac{1}{56}(r^2+6)H(ni3333)+\tfrac{1}{28}r_nH(i3333) \\ +\tfrac{1}{56}(12r^2+23)r_iH(n3333)+\tfrac{1}{56}(12r^2+23)\delta_{in}H(3333)+\tfrac{3}{7}r_nr_iH(3333) \\ +\tfrac{1}{2}\{\delta_{i3}H(n333)+r_3H(ni333)+\delta_{3n}H(i333)+3r_ir_3H(n333) \\ +3\delta_{in}r_3H(333)+3\delta_{3n}r_iH(333)\}] \\ \times[\delta_{1i}(1-\tfrac{3}{4}r^{-1}-\tfrac{1}{4}r^{-3})-r_1r_i\tfrac{3}{4}(r^{-3}-r^{-5})] \\ \times[\delta_{1n}(1-\tfrac{3}{4}r^{-1}-\tfrac{1}{4}r^{-3})-r_1r_n\tfrac{3}{4}(r^{-3}-r^{-5})]\,dV. \quad \text{(A 2.22)}$$

By using the lemmas 1 to 4 this finally gives

$$\int_\Gamma (v_{s1})_i\,(e_{p3})_{in}\,(v_{s1})_n\,dV = -\tfrac{231}{256}J. \quad \text{(A 2.23)}$$

Substituting the results (A 2.18), (A 2.21) and (A 2.23) in the equation (A 2.17), we obtain

$$A_{311} = J(\tfrac{3}{2}-\tfrac{81}{64}+\tfrac{231}{256})\,\epsilon+O(\epsilon^2) = \tfrac{291}{256}J\epsilon+O(\epsilon^2). \quad \text{(A 2.24)}$$

It may be easily proved that

$$J \equiv \int_{S_L} \frac{r_1^2r_3}{r}\frac{\partial^3}{\partial r_3^3}\left(\frac{1}{r}\right)dS = \frac{24\pi}{35}. \quad \text{(A 2.25)}$$

Hence
$$A_{311} = \tfrac{873}{1120}\pi\epsilon+O(\epsilon^2). \quad \text{(A 2.26)}$$

By taking $\mathbf{U} = (U, 0, 0)$ we see that equation (7.32) gives

$$({}^2F)_3 = \mu cURA_{311} = \mu cUR\{\tfrac{873}{1120}\pi\epsilon+O(\epsilon^2)\}. \quad \text{(A 2.27)}$$

Hence we have proved the statement given at the end of § 8 that the contribution ${}^2\mathbf{F}$ to the vector force on a body is not necessarily zero.

In a manner similar to that used above, we will now calculate $\bar{A}_{311}$ for the same deformed sphere. From equation (9.16)

$$\bar{A}_{311} = \int_{S_L} ({}_{-2}v_3)_1\,dS_1 - \int_\Gamma (v_1)_i\,(v_3)_{i,1}\,dV \\ = \epsilon\left\{\int_{S_L} ({}_{-2}v_{p3})_1\,dS_1 - \int_\Gamma (v_{p1})_i\,(v_{s3})_{i,1}\,dV - \int_\Gamma (v_{s1})_i\,(v_{p3})_{i,1}\,dV\right\}+O(\epsilon^2), \quad \text{(A 2.28)}$$

where Γ may now be taken to be the volume exterior to the *sphere* $r = 1$. As before,

$$\int_{S_L} ({}_{-2}v_{p3})_1\,dS_1 = +\tfrac{3}{2}J. \quad \text{(A 2.29)}$$

The values of the two remaining integrals in equation (A 2.28) may be evaluated by the use of lemmas 1 to 4 to give

$$\int_\Gamma (v_{p1})_i\,(v_{s3})_{i,1}\,dV = +\tfrac{9}{4}J, \quad \text{(A 2.30)}$$

$$\int_\Gamma (v_{p3})_{i,1}\,(v_{s1})_i\,dV = -\tfrac{3}{4}J. \quad \text{(A 2.31)}$$

Substituting equations (A 2.29), (A 2.30) and (A 2.31) into (A 2.28), we obtain

$$\bar{A}_{311} = \epsilon J(\tfrac{3}{2} - \tfrac{9}{4} + \tfrac{3}{4}) + O(\epsilon^2) = O(\epsilon^2). \qquad \text{(A 2.32)}$$

Again by taking $\mathbf{U} = (U, 0, 0)$ we see that equation (9.16) gives

$$({}^2\bar{F})_3 = \mu c U R \bar{A}_{311} = \mu c U R\{O(\epsilon^2)\}. \qquad \text{(A 2.33)}$$

Combining this result with (A 2.27) gives

$$({}^2F)_3 - ({}^2\bar{F})_3 = \mu c U R\{\tfrac{873}{1120}\pi\epsilon + O(\epsilon^2)\}. \qquad \text{(A 2.34)}$$

Hence we have proved the statement given at the end of § 9 that ${}^2\mathbf{F}$ and ${}^2\bar{\mathbf{F}}$ are not necessarily the same. Thus for bodies not possessing any of the symmetry properties given in § 8, Oseen's method gives, in general, an incorrect value for the force on the body to $O(R)$.

The authors have shown by considering a different example that, as one would expect, ${}^2\bar{\mathbf{F}}$ is in general non-zero. The proof of this result is not given here, since for bodies not possessing any of the symmetry properties of § 8 for which ${}^2\bar{\mathbf{F}}$ might be non-zero, we cannot use Oseen's method for calculating the force on the body to $O(R)$.

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Note added in proof

The geometric symmetry conditions tabulated in equation (8.2), leading to the relation $\mathbf{A} = 0$ and thus to the vanishing of the 'irreversible' force vector ${}^2\mathbf{F}$, are not the only ones which can give rise to this condition. For example it can be shown that $\mathbf{A} = 0$ for all bodies possessing two axes of helicoidal symmetry‡ intersecting at right angles. As a special case this result shows, incidentally, that equation (8.2*b*) may be modified so as only to require that the body should transform into itself under rotation through $\tfrac{1}{2}\pi$ about each of *two* (rather than *three*) mutually perpendicular axes in order that $\mathbf{A}$ may vanish.

‡ A body possesses an axis of helicoidal symmetry, say the OX_1 axis, if it retains the same relations to the OX_2 and OX_3 axes when the latter are rotated about the OX_1 axis through a definite angle in *either* direction, the angle *not* being equal to 0, $\tfrac{1}{2}\pi$ or π.